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Research Article

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# Application of the Variational Iteration Method to the Initial Value Problems of *Q*-difference Equations-Some Examples

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**Abstract**

The $q$-difference equations are a class of important models both in $q$-calculus and applied sciences. The variational iteration method is extended to approximately solve the initial value problems of $q$-difference equations. A $q$-analogue of the Lagrange multiplier is presented and three examples are illustrated to show the method's efficiency.

**Keywords:** Quantum calculus,Variational iteration method, $q$-derivative

## 1 Introduction

The $q$-calculus has a lot of potential applications in various mathematical areas and appears connections between mathematics and physics, i.e., statistic physics [1], fractal geometry [2, 3] and quantum mechanics. Thus, the letter $q$ has the following meanings [4]: (a) the first letter of "quantum"; (b) the letter commonly used to denote the number of elements in a finite field; (c) the indeterminate of power series.

In the past ten years, some $q$- difference equations have been proposed with linear and nonlinear initial value problems [5]. Naturally, to find the approximate solutions of the $q$-difference equations is under taking. The variational iteration method (VIM) [6, 7] is one of the analytical methods used most often in various initial boundary value problems of differential equations. In this study, the VIM is extended to $q$-differential equations. The obtained solutions containing the parameter $q$ can reduce to the approximate solutions for the initial value problems in ordinary calculus for $q \to 1$.

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## 2 Properties of $q$-calculus and preliminaries of the VIM

For $0<q<1$, let $T_q$ be the time scale

$T_q=\{q^m : m\in \mathrm{Z}\}\cup\{0\}$

where $\mathrm{Z}$ is the set of the integers.

### 2.1 *Q*-calculus

Let $f(x;y;...)$ be a multivariable real continuous function. The partial $q$-derivative is defined by Jackson [8]

$$\frac{\partial_q}{\partial_q x} f(x;y;...)=\frac{f(qx;y;...)-f(x;y;...)}{(q-1)x}, x\in T_q, \tag{1}$$

and

$$\frac{\partial_q}{\partial_q x} f(x;y;...)|_{x=0}=\lim_{n\to\infty}\frac{f(q^n;y;...)-f(0;y;...)}{q^n}.$$

The corresponding $q$-integral [9] is

$$\int_0^x f(t)d_q t=(1-q)x\sum_0^{\infty} q^n g(q^n x). \tag{2}$$

### 2.2 Q -Leibniz product law is

$$D_q[g(x)f(x)]=g(qx)D_q[f(x)]+f(x)D_q[g(x)]. \tag{3}$$

where $D_q$ is the *q*-derivative.

More generally, one can have

$$D_q^n[g(x)f(x)]=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q D_q^{n-k} f(q^k x)D_q^k g(x). \tag{4}$$

For each positive integer $n$, the $q-$integer $[k]_q$ and the $q-$factorial $[k]_q!$ [4] are defined by

$$[k]_q=\begin{cases} 1+q+...+q^{k-1}, q\neq 1, \\ k,\ \ q=1, \end{cases}$$

$$[k]_q!=\begin{cases} [1]_q[2]_q...[k]_q, k\geq 1, \\ 1, k=0, \end{cases}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q=\frac{[n]_q!}{[k]_q![n-k]_q!}.$$

**2.3 Q-Integration by parts [4] holds**

$$\int_a^b g(qt)D_q[f(t)]d_qt = [f(t)g(t)]\Big/_a^b - \int_a^b f(t)D_q[g(t)]d_qt. \tag{5}$$

The properties above are needed when constructing the correction functional of the $q$-difference equations. More recent development in this area, i.e., the modified $q$-derivative, the Laplace transform, the Taylor theorem and applications in quantum mechanics can be found in [4, 10, 11-18].

**2.4 To illustrate its basic idea of the VIM [6, 7], consider the following difference equation:**

$$\frac{d^m u}{dt^m} + R[u(t)] + N[u(t)] = g(t), \tag{6}$$

where $R$ is a linear operator, $N$ is a nonlinear operator, $g(t)$ is a given continuous function and $\frac{d^m u}{dt^m}$ is the term of the highest-order derivative. The basic character of the method is to construct a correction functional for Eq. (6)

$$u_{n+1} = u_n + \int_0^t \lambda(t,s)(\frac{d^m u_n}{ds^m} + R[u_n] + N[u_n] - g(s))ds. \tag{7}$$

Here the unknown function $\lambda(t,s)$ is called a Lagrange multiplier which can be identified optimally via the variational theory and $u_n$ is the $n$-th term approximate solution. Eq. (6) has a Lagrange multiplier

$$\lambda = \frac{(-1)^m (s-t)^{m-1}}{(m-1)!} \tag{8}$$

and the corresponding iteration formula is

$$u_{n+1} = u_n + \int_0^t \frac{(-1)^m (s-t)^{m-1}}{(m-1)!}(\frac{d^m u_n(s)}{ds^m} + R[u_n(s)] + N[u_n(s)] - g(s))ds, \tag{9}$$

where the initial iteration $u_0$ can be determined by Taylor theorem. The method is extended to the $q$-difference equations of second order [19] and third order [20], respectively. Let us consider some applications.

## 3 Approximate solutions of $q$-difference equations

**Example 1:** Consider the following $q$-diffusion equation [8]

$$\frac{\partial_q}{\partial_q t} u(x,t) = \frac{\partial_q^2}{\partial_q x^2} u(x,t), u(x,0) = e_q^x. \tag{10}$$

The model above allows the $q$-exponential distribution which describes a non-equilibrium system.
We construct the correction functional through Jackson's integral

$$u_{n+1} = u_n + \int_0^t \lambda(t, q\tau)[\frac{\partial_q}{\partial_q \tau} u_n(x,\tau) - \frac{\partial_q^2}{\partial_q x^2} u_n(x,\tau)]d_q\tau. \tag{11}$$

The corrected functional here is different from the one in ordinary since the parameter $q$ disappears after the integration by parts (5). Thus, we consider using $\lambda(t, q\tau)$ in the functional (11).

Considering $\frac{\partial_q^2}{\partial_q x^2} u_n(x,\tau)$ as a restricted variation and taking the variation derivative on the both sides of Eq. (11) results in

$$\begin{aligned}\delta u_{n+1} &= \delta u_n + \delta\int_0^t \lambda(t,q\tau)[\frac{\partial_q}{\partial_q \tau} u_n(x,\tau)]d_q\tau \\ &= (1+\lambda(t,\tau)/_{\tau=t})\delta u_n - \int_0^t [\frac{\partial_q}{\partial_q\tau}\lambda(t,\tau)]\delta u_n(x,\tau)d_q\tau.\end{aligned} \tag{12}$$

The extreme condition $\delta u_{n+1} = 0$ requires the system

$$1+\lambda(t,\tau)/_{\tau=t} = 0, \frac{\partial_q}{\partial_q \tau}\lambda(t,\tau) = 0. \tag{13}$$

As a result, the $q$-Lagrange multiplier can be identified as

$$\lambda(t,\tau) = -1. \tag{14}$$

Now we determine the iteration formula (11) as

$$u_{n+1} = u_n - \int_0^t [\frac{\partial_q}{\partial_q\tau} u_n(x,\tau) - \frac{\partial_q^2}{\partial_q x^2} u_n(x,\tau)]d_q\tau. \tag{15}$$

The initial iteration is assumed as

$$u_0 = u(x,0) = e_q^x.$$

The successive approximate solutions can be obtained as

$$u_1 = e_q^x(1+\frac{t}{[1]_q!}),$$

$$u_2 = e_q^x(1+\frac{t}{[1]_q!}+\frac{t^2}{[2]_q!}),$$

$$\vdots$$

$$u_n = e_q^x(1+\frac{t}{[1]_q!}+\frac{t^2}{[2]_q!}+\ldots+\frac{t^n}{[n]_q!}).$$

$u_n$ tends to $e_q^x\, e_q^t$ for $n\to\infty$, which is an exact solution of Eq. (1). The same result can be obtained by the $q$-difference transform [11]. Different $q$ can be chosen in the exponential distribution function to describe the non-equilibrium system.

**Example 2:** Consider a nonlinear $q$-difference equation

$$D_q f(t) = -f^2(t)+1,\ f(0)=0. \tag{16}$$

For $q=1$, the Riccati equation

$$\frac{df(t)}{dt} = -f^2(t)+1,\quad f(0)=0, \tag{17}$$

has the exact solution

$f(t)=\tanh(t).$

We can have the iteration formula for Eq. (16)

$$\begin{cases} f_{n+1}=f_n-\int_0^t [D_q f_n(x,\tau)+f_n^{\,2}(x,\tau)-1]d_q\tau, \\ f_0=0. \end{cases}$$

As a result, we can obtain the series solution as

$$f_0=0,$$
$$f_1=\frac{t}{[1]_q},$$
$$f_2=\frac{t}{[1]_q}-\frac{t^3}{[3]_q},$$
$$f_3=\frac{t}{[1]_q}-\frac{t^3}{[3]_q}+\frac{2t^5}{[3]_q[5]_q}-\frac{t^7}{[3]_q^2[7]_q},$$
$$\ldots\ldots$$

For $q\to 1$, $f_3$ tends to the variational iteration solution for the initial value problem of Eq. (17).

In the above derivation, one can decompose the nonlinear term with the linearized techniques and also use the Pade-technique to accelerate the convergence. For the other development of the VIM, readers are referred to the method using the fractional differential equations [21], a new approach to identification of the Lagrange multipliers [22], applications to fuzzy equations [23], new Lagrange multipliers of the VIM in fractional calculus [24-26] and applications in the eight-order boundary value problem [27], integro-differential equation [28] and the wave-diffusion equation [29].

**Example 3:** Now consider the $q$-analogue of an oscillator equation of second order

$$D_q^2 u-u=0,\ u(0)=1,\ D_q u\big/_{t=0}=1. \tag{18}$$

Construct the correction the functional

$$u_{n+1}=u_n+\int_0^t \lambda(t,q^2\tau)[D_q^2 f_n(x,\tau)u_n(\tau)-u_n(\tau)]d_q\tau. \tag{19}$$

Through the integration by parts and considering the term $u_n(\tau)$ as a restricted variation, we get

$$\delta u_{n+1}=(1-q\frac{\partial_q}{\partial_q\tau}\lambda(t,\tau)\big/_{\tau=t})\delta u_n+\lambda(t,q\tau)\big/_{\tau=t}\delta u_n^{'}(x,\tau)-\int_0^t \delta u_n\frac{\partial_q^2}{\partial_q\tau^2}\lambda(t,\tau)d\tau, \tag{20}$$

where $\delta$ denotes the variation derivative and“ ' ” denotes the derivative.

As a result, the system of the Lagrange multiplier can be obtained

$$\begin{cases} 1-q\dfrac{\partial_q}{\partial_q\tau}\lambda(t,\tau)\Big/_{\tau=t}=0, \\ \lambda(t,q\tau)\Big/_{\tau=t}=0, \\ \dfrac{\partial_q^2}{\partial_q\tau^2}\lambda(t,\tau)=0, \end{cases}$$

from which we can identify

$$\lambda(t,\tau)=q^{-1}(\tau-tq). \tag{21}$$

Substituting Eq. (21) into Eq. (19), the iteration formula can be determined

$$\begin{cases} u_{n+1}=u_n+\dfrac{1}{q}\displaystyle\int_0^t(q^2\tau-qt)[D_q^2u_n(\tau)-u_n(\tau)]d_q\tau, \\ u_0=1+\dfrac{t}{[1]_q!}. \end{cases}$$

The successive solution can be obtained as

$$u_0=1+t=1+\frac{t}{[1]_q!},$$

$$u_1=1+\frac{t}{[1]_q!}+\frac{1}{q}\int_0^t(q^2\tau-qt)(-1-\frac{\tau}{[1]_q!})d_q\tau$$

$$=1+t+[(\tau-t)(-\frac{\tau}{[1]_q!}-\frac{\tau^2}{[2]_q!})]\Big/_0^t-\int_0^t(-\frac{\tau}{[1]_q!}-\frac{\tau^2}{[2]_q!})d_q\tau]$$

$$=1+\frac{t}{[1]_q!}+\frac{t^2}{[2]_q!}+\frac{t^3}{[3]_q!},$$

...

$$u_n=\sum_{k=0}^{2n+1}\frac{t^k}{[k]_q!}.$$

Recall that the limit $u=\lim\limits_{t\to\infty}u_n=e_q^t$ is an exact solution of Eq. (18) which shows the validness of the Lagrange multiplier (21).

## 4 Conclusions

In this study, the VIM for solving the initial value problems of the q-difference equations is proposed. In the method, one of the crucial steps is to identify the Lagrange multipliers. This study suggests the basic principles in calculating a q-analogue Lagrange multipliers. Three examples from linear to nonlinear ones, first order to second order are illustrated and the exact solutions are used to check the variational iteration formulae's validness. Now other nonlinear techniques coupled with the VIM become possible. The presented method also can be applied to other difference equations in quantum calculus.

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