Communications in Numerical Analysis 2013 (2013) 1-5

ISPACS International Scientific Publications and Consulting Services

Available online at www.ispacs.com/cna
Volume 2013, Year 2013 Article ID cna-00173, 5 Pages
doi:10.5899/2013/cna-00173
Research Article

Communications in Numerical Analysis

Traveling wave solutions of a biological reaction-convection-diffusion equation model by using (G'/G) expansion method

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Abstract

In this paper, the (G'/G)-expansion method is applied to solve a biological reaction-convection-diffusion model arising in mathematical biology. Exact traveling wave solutions are obtained by this method. This scheme can be applied to a wide class of nonlinear partial differential equations.

Keywords: Expansion methods, Reaction-convection-diffusion equation, Nonlinear evolution equations, Exact Solutions

1 Introduction

Mathematical modeling of physical and biological systems often leads to nonlinear evolution equations. Exact solutions of these equations are of theoretical importance. Considerable efforts have been made to the study of solitary wave solutions. Various ansatze have been proposed for seeking traveling wave solutions of nonlinear differential equations. Recently, Many new methods have been proposed to find some particular solutions for these problems for instance, the Exp-function method [1, 2, 3], the homogeneous balance method [4, 5], the sine-cosine method [6], the rational expansion method and it's generalizations [7, 8, 9], the Jacobi elliptic function method [10], the F-expansion method and its extensions [11, 12, 13]. The objective of this paper is to use the (G'/G)-expansion method to investigate exact traveling wave solutions of a biological model that is a reaction-convection-diffusion equation, namely Murray equation which can be considered as a generalization of the Fisher and Burgers equations. The (G'/G)-expansion method is based on the explicit linearization of nonlinear evolution equations for traveling waves with a certain substitution which leads to a second-order differential equation with constant coefficients [17-22]. Through the use of the method we can obtain more general solutions with some free parameters. Moreover, the (G'/G)-expansion method transforms a nonlinear equation to a simple algebraic system of equations which can be solved easily by means of a symbolic computational software like Maple, Matlab, Mathematica, etc. In this paper, we use Maple 15.

2 Description of the (G'/G)-expansion method

Wang et al. [15] summarized the main steps for using the (G'/G)-expansion method, as follows:

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Step 1. The traveling wave variable $\xi = kx + wt$ permits us to reduce the following partial differential equation

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \dots) = 0, \tag{2.1}$$

to an ordinary differential equation for $U = U(\xi)$ of the form

$$Q(U, -wU', kU', w^2U'', k^2U'', kwU'', \dots) = 0. \tag{2.2}$$

After solving the ODE (2.2), we have the some traveling wave solutions $u(x,t) = U(\xi)$ of the PDE (2.1).

Step 2. Suppose that the solution of the ODE (2.2) can be expressed as a polynomial in (G'/G) as follows:

$$U(\xi) = \sum_{i=0}^{n} \alpha_i (\frac{G'}{G})^i, \tag{2.3}$$

where $G = G(\xi)$ satisfies the second-order linear ODE in the form

$$G'' + \lambda G' + \mu G = 0, \tag{2.4}$$

and α_i, $i = 0, \dots, n$, λ and μ are constants to be determined and the leading coefficient α_n is nonzero. The positive integer n can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (2.2).

Step 3. Substituting (2.3) into (2.2) and using (2.4), and then collecting all terms with the same power of (G'/G) together, the left-hand side of Eq. (2.2) is converted into a polynomial in (G'/G). By equating each coefficient of the resulted polynomial to zero, a system of algebraic equations is obtained for α_i's, k, w, λ, μ that can be solved by a computational algebraic system (CAS) like Maple.

Step 4. Since the general solutions of the ODE (2.4) are well known for us, by substituting α_i's, k, w and the general solutions of Eq. (2.4) into Eq. (2.3), we have more traveling wave solutions of the nonlinear evolution equation (2.1).

Remark 2.1. *The function $G(\xi)$ which is the solution of Eq. (2.4) has the property that different order derivatives of the function (G'/G) can be expressed as a second order polynomial with respect to (G'/G). In fact, using (2.4) we have*

$$\frac{d(\frac{G'}{G})}{d\xi} = -(\mu + \lambda(\frac{G'}{G}) + (\frac{G'}{G})^2), \tag{2.5}$$

and so we have the following derivatives of $U(\xi)$

$$U' = \frac{dU}{d\xi} = -(\mu + \lambda(\frac{G'}{G}) + (\frac{G'}{G})^2)\frac{dU}{d(\frac{G'}{G})}, \tag{2.6}$$

$$U'' = \frac{d^2U}{d\xi^2} = (\mu + \lambda(\frac{G'}{G}) + (\frac{G'}{G})^2)^2 \frac{d^2U}{d(\frac{G'}{G})^2} + (\lambda + 2(\frac{G'}{G}))(\mu + \lambda, (\frac{G'}{G}) + (\frac{G'}{G})^2)\frac{dU}{d(\frac{G'}{G})}, \tag{2.7}$$

and etc. So by considering $U(\xi)$ as Eq. (2.3), it follows that its derivatives are polynomials of (G'/G).

Remark 2.2. *The general solution $G(\xi)$ of Eq. (2.4) required in the last step has an explicit form and so the expression G'/G is as follows*

$$\frac{G'(\xi)}{G(\xi)} = \begin{cases} \frac{\sqrt{\lambda^2-4\mu}}{2}(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}) - \frac{\lambda}{2}, & \lambda^2 - 4\mu > 0, \\ \frac{\sqrt{4\mu-\lambda^2}}{2}(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}) - \frac{\lambda}{2}, & \lambda^2 - 4\mu < 0. \end{cases} \tag{2.8}$$

3 Application of the method

Consider the following reaction-convection-diffusion equation of the form

$$u_t = (\lambda + \lambda_0 u)u_{xx} + \lambda_1 u u_x + \lambda_2 u - \lambda_3 u^2, \tag{3.9}$$

where λ, λ_0, λ_1, λ_2 and λ_3 are real constants [20]. In the particular case $\lambda = 1$ and $\lambda_0 = 0$, this equation coincides with the Murray equation

$$u_t = u_{xx} + \lambda_1 u u_x + \lambda_2 u - \lambda_3 u^2, \tag{3.10}$$

which itself is a generalization of the well-known Fisher equation when $\lambda_1 = 0$. When both λ_2 and λ_3 are zero, it is reduced to the classical Burgers equation. We introduce the traveling wave variable $u(x,t) = U(\xi)$, $\xi = kx + wt$ into Eq. (3.10) to find

$$wU' - k^2U'' - \lambda_1 kUU' - \lambda_2 U + \lambda_3 U^2 = 0, \tag{3.11}$$

where prime denotes the derivatives with respect to ξ. Considering the homogeneous balance between the highest linear term U'' and the nonlinear term UU' in Eq. (3.11), the parameter n that is required in Eq. (2.3) is determined. In fact by using (2.7), we have $n+2 = 2n+1$ and so $n = 1$. Therefore, the solution can be expressed as

$$U(\xi) = \alpha_0 + \alpha_1(G'/G), \tag{3.12}$$

where α_0 and α_1 are constants to be determined later. Substituting Eq. (3.12) along with (2.6) and (2.7) into Eq. (3.11) and collecting all terms with the same power of (G'/G) together, the left hand side of Eq. (3.11) is converted into a polynomial in (G'/G). Equating each coefficient to be zero yields a set of simultaneous algebraic equations. Solving this system by Maple, we get some trivial and nontrivial solutions. The trivial obtained solutions are $U = 0, \frac{\lambda_2}{\lambda_3}$ and the nontrivial traveling wave solution is as follows:

$$\begin{aligned} k &= \pm\frac{\lambda_1\lambda_2}{2\lambda_3\sqrt{\lambda^2-4\mu}}, \quad w = \pm\frac{\lambda_2(\lambda_1^2\lambda_2+4\lambda_3^2)}{4\lambda_3^2\sqrt{\lambda^2-4\mu}}, \\ \alpha_0 &= \frac{\lambda_2}{2\lambda_3}(1\pm\frac{\lambda}{\sqrt{\lambda^2-4\mu}}), \quad \alpha_1 = \pm\frac{\lambda_2}{\lambda_3\sqrt{\lambda^2-4\mu}}. \end{aligned} \tag{3.13}$$

Case 1. When $\lambda^2 - 4\mu > 0$, by substituting (3.13) into Eq. (3.12) and using (2.8), the hyperbolic solutions are obtained as follows:

$$\begin{aligned} U^{\pm}_{hyper}(\xi) &= \frac{\lambda_2}{2\lambda_3}(1\pm\frac{\lambda}{\sqrt{\lambda^2-4\mu}}) \\ &\pm\frac{\lambda_2}{\lambda_3\sqrt{\lambda^2-4\mu}}(\frac{\sqrt{\lambda^2-4\mu}}{2}(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})-\frac{\lambda}{2}), \end{aligned} \tag{3.14}$$

where $\xi = (\pm\frac{\lambda_1\lambda_2}{2\lambda_3\sqrt{\lambda^2-4\mu}})x + (\pm\frac{\lambda_2(\lambda_1^2\lambda_2+4\lambda_3^2)}{4\lambda_3^2\sqrt{\lambda^2-4\mu}})t$ and C_1, C_2 are arbitrary constants. So we get different solutions $u(x,t) = U(\xi)$.

Case 2. When $\lambda^2 - 4\mu < 0$ by substituting (3.13) along with (2.8) into (3.12), and then simplifying the resulted solution, we have the following trigonometric form

$$\begin{aligned} U^{\pm}_{trig}(\xi) &= \frac{\lambda_2}{2\lambda_3}(1\pm\frac{\lambda}{\sqrt{\lambda^2-4\mu}}) \\ &\pm\frac{\lambda_2}{\lambda_3\sqrt{\lambda^2-4\mu}}(\frac{\sqrt{4\mu-\lambda^2}}{2}(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi})-\frac{\lambda}{2}), \end{aligned} \tag{3.15}$$

where the traveling wave is as

$$\xi = \pm\frac{\lambda_1\lambda_2}{2\lambda_3\sqrt{\lambda^2-4\mu}}x \pm \frac{\lambda_2(\lambda_1^2\lambda_2+4\lambda_3^2)}{4\lambda_3^2\sqrt{\lambda^2-4\mu}}t,$$

and C_1, C_2 are arbitrary constants. So we get different traveling wave solutions $u(x,t) = U(\xi)$ also in this case.

Remark 3.1. *All of the derived solutions are tested by direct substitution in the studied problems to ensure the correctness.*

4 Conclusions

Traveling wave solutions to nonlinear evolution equations arising in mathematical biological systems are of theoretical importance. Various ansatze have been proposed for seeking traveling wave solutions of nonlinear differential equations. In this paper, the (G'/G) expansion method is successfully applied for obtaining exact traveling wave solutions of a reaction-convection-diffusion equation (i.e., Murray's biological model). It is shown that the (G'/G)-expansion method is quiet efficient and well suited for finding exact solutions. The reliability of the method and reduction in the size of computational domain give this method a wider applicability. With the aid of Maple and by putting them back into the original equation, we have assured the correctness of the obtained solutions. Finally, it is worthwhile to mention that the method is straightforward and concise and it can be applied to other nonlinear evolution equations in engineering and the physical sciences.

Acknowledgements

The authors would like to thank the referee for his/her valuable comments.

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