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Research Article

Minimal solution for inconsistent singular fuzzy matrix equations

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Abstract

The fuzzy matrix equations $A\tilde{X} = \tilde{Y}$ is called a singular fuzzy matrix equations while the coefficients matrix of its equivalent crisp matrix equations be a singular matrix. The singular fuzzy matrix equations are divided into two parts: consistent singular matrix equations and inconsistent fuzzy matrix equations. In this paper, the inconsistent singular fuzzy matrix equations is studied and the effect of generalized inverses in finding minimal solution of an inconsistent singular fuzzy matrix equations are investigated.

Keywords: Crisp system, Generalized inverse, Singular system, Fuzzy matrix equations.

1 Introduction

The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh [10, 25]. The importance of the introduced notion of fuzzy set was realized and has successfully been applied in almost all the branches of science and technology. Recently fuzzy set theory has been applied in pure mathematics by Tripathy and Baruah [15], Tripathy and Borgohain [16], Tripathy, Sen and Nath [17], Tripathy and Das [18], Tripathy and Sarma [19], Tripathy, Baruah, Et and Gungor [20], Tripathy and Ray [21] and many others. A $n \times n$ linear system whose coefficient matrix is crisp and the right hand side column is an arbitrary fuzzy number vector, is called a fuzzy linear system. Friedman et al. introduced a general model for solving fuzzy linear systems [12]. A method for solving nonlinear matrix equation was given in [11]. Solving fuzzy linear systems have been studied by many authors [3, 4, 6, 13]. In [7] the original $n \times n$ fuzzy linear system with a nonsingular matrix A is replaced by two $n \times n$ crisp linear systems. In recent years needs have been felt in numerous areas of applied mathematics and fuzzy mathematics for some kind of generalized inverse of a matrix that is singular or even rectangular [8, 23, 26]. There are different generalized inverses for different purposes [8, 9, 14]. On the least-square solution for inconsistent fuzzy matrix equations using $\{1,3\}$-inverse, one of the generalized inverses, was discussed in [26]. Wang et al. studied on fuzzy least squares solutions for inconsistent fuzzy linear systems, using $\{1,3\}$-inverse and pseudoinverse [24].

In this paper, $\{1\}$-inverse, Drazin inverse and pseudoinverse in finding minimal solution of an inconsistent singular fuzzy matrix equations are used. Section 2 gives preliminaries. The effect of generalized inverses in finding minimal solution of an inconsistent singular fuzzy matrix equations are investigated in section 3. Numerical examples are given in section 4 followed by a suggestion and concluding remarks in section 5.

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2 Preliminaries and basic definitions

In this section, the following definitions and basic results is reviewed. For more details, we refer the reader to [1, 2, 22, 23].

Definition 2.1. *Let $A \in C^{n\times n}$. The index of matrix A is equivalent to the dimension of largest Jordan block corresponding to the zero eigenvalue of A and is denoted by $ind(A)$.*

Definition 2.2. *Let $A \in C^{n\times n}$, with $ind(A) = k$. The matrix X of order n is the Drazin inverse of A, denoted by A^D, if X satisfies the following conditions*

$$AX = XA,\ XAX = X,\ A^kXA = A^k,$$

when $ind(A) = 1$, A^D is called the group inverse of A, and denoted by A_g.

Theorem 2.1. *([9, 22]) Let $A \in C^{n\times n}$, with $ind(A) = k$, $rank(A^k) = r$. We may assume that the Jordan normal form of A has the form as follows*

$$A = P\begin{pmatrix} D & 0 \\ 0 & N \end{pmatrix}P^{-1},$$

where P is a nonsingular matrix, D is a nonsingular matrix of order r, and N is a nilpotent matrix that $N^k = \bar{o}$. Then we can write the Drazin inverse of A in the form

$$A^D = P\begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix}P^{-1},$$

when $ind(A) = 1$, obviously, $N = \bar{o}$.

Theorem 2.2. *([22]) For any matrix $A \in C^{n\times n}$ the index and Drazin inverse of A exists and is unique.*

Theorem 2.3. *([9]) A^Db is a solution of*

$$Ax = b,\ \ k = ind(A), \tag{2.1}$$

if and only if $b \in R(A^k)$, and A^Db is an unique solution of (2.1) provided that $x \in R(A^k)$.

Definition 2.3. *Let $A \in C^{m\times n}$. The matrix X of order $n\times m$ is the pseudoinverse of A denoted by A^+, if X satisfies the following conditions*

$$AXA = A,\ \ XAX = X,\ \ (AX)^* = AX,\ \ (XA)^* = XA.$$

Theorem 2.4. *([8]) Let $A \in C^{m\times n}$. We may assume that the singular value decomposition of A has the form as follows*

$$A = P\begin{pmatrix} D & 0 \\ 0 & N \end{pmatrix}Q^*,$$

where P is an $m\times m$ unitary matrix, D is an $m\times n$ diagonal matrix, and Q is an $n\times n$ unitary matrix, then putting

$$A^+ = Q^*\begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix}P^*.$$

Definition 2.4. *Let $A \in C^{m\times n}$. The matrix X of order $n\times m$ is the $\{1\}$-inverse of A denoted by $A^{(1)}$, if X satisfies the following condition*

$$AXA = A.$$

Theorem 2.5. *([8]) Let $A \in C_r^{m\times n}$, and let $E \in C_m^{m\times m}$ and $P \in C_n^{n\times n}$ be such that*

$$EAP = \begin{pmatrix} I_r & K \\ 0 & 0 \end{pmatrix}.$$

Then for any $L \in C^{(n-r)\times(m-r)}$, the $m\times n$ matrix

$$X = P\begin{pmatrix} I_r & 0 \\ 0 & L \end{pmatrix}E,$$

is a $\{1\}-$inverse of A.

Definition 2.5. *([14]) Consider a system of equations written in matrix form as $Ax = b$ where A is $m \times n$, x is $n \times 1$, and b is $m \times 1$. The minimal solution of this problem is defined as follows:*
1. If the system is consistent and has a unique solution, x, then the minimal solution is defined to be x.
2. If the system is consistent and has a set of solutions, then the minimal solution is the element of this set having the least Euclidean norm.
3. If the system is inconsistent and has a unique least-squares solution, x, the minimal solution is defined to be x.
4. If the system is inconsistent and has set of least-squares solutions, then the minimal solution is the element of this set having the least Euclidean norm.

Theorem 2.6. *([14]) The minimal solution of the system*

$$Ax = b,$$

is given by the pseudoinverse $x = A^{+}b$.

Definition 2.6. *The set of all these fuzzy numbers in parametric form is denoted by E. A fuzzy number $\widetilde{u}$ in parametric form is a pair $(\bar{u}(r), \underline{u}(r))$ of functions $\bar{u}(r), \underline{u}(r), 0 \le r \le 1$, which satisfy the following requirements*
1. $\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0,1]$,
2. $\bar{u}(r)$ is a bounded left continuous non-increasing function over $[0,1]$,
3. $\underline{u}(r) \le \bar{u}(r), 0 \le r \le 1$.

Definition 2.7. *For arbitrary fuzzy numbers $\widetilde{x} = (\underline{x}(r), \bar{x}(r))$, $\widetilde{y} = (\underline{y}(r), \bar{y}(r))$ and $k \in R$, we may define the addition and the scalar multiplication of fuzzy numbers as*
1. $\widetilde{x} + \widetilde{y} = (\underline{x}(r) + \underline{y}(r), \bar{x}(r) + \bar{y}(r))$,
2. $k\widetilde{x} = \begin{cases} (k\underline{x}(r), k\bar{x}(r)), & k \ge 0, \\ (k\bar{x}(r), k\underline{x}(r)), & k < 0. \end{cases}$

Definition 2.8. *The following fuzzy matrix equations*

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \tilde{x}_{11} & \cdots & \tilde{x}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{x}_{n1} & \cdots & \tilde{x}_{nn} \end{pmatrix} = \begin{pmatrix} \tilde{y}_{11} & \cdots & \tilde{y}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{y}_{n1} & \cdots & \tilde{y}_{nn} \end{pmatrix}, \tag{2.2}$$

where $A = (a_{ij}),\ \ 1 \le i, j \le n$ is a real matrix, and the elements $(\tilde{y}_{ij}),\ \ 1 \le i, j \le n$ in the right-hand side matrix are fuzzy numbers can be extended into the crisp matrix equation

$$SX(r) = Y(r), \tag{2.3}$$

where $S = (s_{ij}), 1 \le i \le 2n$, $1 \le j \le 2n$ are determined as follows:

$$\begin{gathered} a_{ij} \ge 0 \Rightarrow s_{ij} = a_{ij}, s_{i+n,j+n} = a_{ij}, \\ a_{ij} < 0 \Rightarrow s_{i,j+n} = -a_{ij}, s_{i+n,j} = -a_{ij}, \end{gathered} \tag{2.4}$$

and any (s_{ij}) which is not determined by (2.4)(2.4) is zero, and

$$x_j(r) = \begin{pmatrix} \underline{x}_j(r) \\ -\bar{x}_j(r) \end{pmatrix} = \begin{pmatrix} \underline{x}_{1j}(r) \\ \vdots \\ \underline{x}_{nj}(r) \\ -\bar{x}_{1j}(r) \\ \vdots \\ -\bar{x}_{nj}(r) \end{pmatrix}, \ \ 1 \le j \le n, \ \ X(r) = \left(x_1(r), \cdots x_n(r)\right),$$

$$y_j(r)=\begin{pmatrix}\underline{y}_j(r)\\ -\bar{y}_j(r)\end{pmatrix}=\begin{pmatrix}\underline{y}_{1j}(r)\\ \vdots\\ \underline{y}_{nj}(r)\\ -\bar{y}_{1j}(r)\\ \vdots\\ -\bar{y}_{nj}(r)\end{pmatrix},\ \ 1\le j\le n,\ \ Y(r)=\big(y_1(r),\cdots y_n(r)\big).$$

The structure of $S=(s_{ij}), 1\le i\le 2n$ and $1\le j\le 2n$ implies that

$$S=\begin{pmatrix}B & C\\ C & B\end{pmatrix},$$

where B contains the positive entries of A and C contains the absolute value of the negative entries of A, i.e., $A=B-C$.

Definition 2.9. *The fuzzy matrix equations (2.2) is called a singular fuzzy matrix equations while its extended crisp matrix equations (2.3) be a singular matrix equations. Hence, the index of the matrix S be nonzero.*

Theorem 2.7. *([26])The fuzzy matrix equations (2.2) is a consistent fuzzy matrix equations, if and only if $rank[S]=rank[S|y_i(r)]\ \ 1\le i\le n$.*

Theorem 2.8. *([8, 23])The extended matrix equations of fuzzy matrix equations $A\tilde{X}=\tilde{Y}$ wherein A is a nonsingular real matrix may be singular matrix equations.*

Definition 2.10. *([5, 12]) Let $X(r)=\{\underline{x}_i(r),-\bar{x}_i(r)), 1\le i\le n\}$ denote a solution of the crisp linear system $SX=Y$. The fuzzy number vector $U=\{\underline{u}_i(r),-\bar{u}_i(r)), 1\le i\le n\}$ defined by*

$$\underline{u}_i(r)=min\{\underline{x}_i(r),\bar{x}_i(r),\underline{x}_i(1),\bar{x}_i(1)\},$$
$$\bar{u}_i(r)=max\{\underline{x}_i(r),\bar{x}_i(r),\underline{x}_i(1),\bar{x}_i(1)\},$$

is called a fuzzy solution of $SX=Y$. If $(\underline{x}_i(r),\bar{x}_i(r)), 1\le i\le n)$, are all fuzzy numbers and $\underline{x}_i(r)=\underline{u}_i(r),\bar{x}_i(r)=\bar{u}_i(r), 1\le i\le n$, then U is called a strong fuzzy solution. Otherwise, U is a weak fuzzy solution.

3 Least-Squares solutions

In this section, generalized inverses in finding minimal solution of

$$SX(r)=Y(r), \tag{3.5}$$

that is the $2n\times 2n$ extended matrix equations of the inconsistent singular fuzzy matrix equations $A\tilde{X}=\tilde{Y}$ are used.

Definition 3.1. *The matrix equations*

$$S^TSX(r)=S^TY(r),\ \ k=ind(S^TS), \tag{3.6}$$

is called the normal equations of the matrix equations (3.5).

Theorem 3.1. *A solution of (3.6) is $X(r)=(S^TS)^DS^TY(r)$.*

Proof. The normal equations of the inconsistent matrix equations (3.5) is a consistent singular matrix equations [14, 23]. The consistent matrix equations (3.6) has a set of solutions. Therefore singular inconsistent matrix equations (3.5) has set of least-square solutions. Drazin inverse for any singular or nonsingular matrix exist and is unique [22]. Campbell [9] showed that

$$X(r)=(S^TS)^DS^TY(r), \tag{3.7}$$

is a solution of (3.6) if and only if $y_i(r)\in R((S^TS)^k),\ \ 1\le i\le n$ and (3.7) is a least-square solution of (3.5). □

Theorem 3.2. *Minimal solution of the matrix equations (3.5) is* $X(r) = (S^T S)^+ S^T Y(r)$.

Proof. Pseudoinverse in finding minimal solution of consistent or inconsistent linear system of equations is used [14]. Any square and rectangular matrix has an unique pseudoinverse [14]. Normal equations for any inconsistent or consistent matrix equations is a consistent singular matrix equations [8, 14]. The matrix equations (3.5) has set of least-squares solutions, then the minimal solution is the element of this set having the least Euclidean norm. From [14, 23] least-squares solution

$$X(r) = (S^T S)^+ S^T Y(r),$$

has least Euclidean norm. □

Theorem 3.3. *A least-square solution of (3.5) is*

$$X(r) = (S^T S)^{(1)}(S^T Y(r)) + (I_{2n} - (S^T S)^{(1)}(S^T S))y, \tag{3.8}$$

for arbitrary $y \in C^{2n\times n}$.

Proof. From [8] (3.8) is a solutions of the singular matrix equations (3.6). Then same as the proof of Theorem 1 in [9] (3.8) is a least-square solution of inconsistent matrix equations (3.5). □

4 Numerical examples

We now give the following example to explain the present results.

Example 4.1. *Consider the following singular fuzzy matrix equations*

$$\begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{pmatrix} = \begin{pmatrix} (2r+1, 5-2r) & (2r+3, 7-2r) \\ (3r+1, 6-2r) & (6+4r, 14-4r) \end{pmatrix}. \tag{4.9}$$

The extended 4×4 *matrix equations of (4.9) is*

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} \underline{x}_{11} & \underline{x}_{12} \\ \underline{x}_{21} & \underline{x}_{22} \\ -\bar{x}_{11} & -\bar{x}_{12} \\ -\bar{x}_{21} & -\bar{x}_{22} \end{pmatrix} = \begin{pmatrix} 2r+1 & 2r+3 \\ 3r+1 & 6+4r \\ 2r-5 & 2r-7 \\ 2r-6 & 4r-14 \end{pmatrix}. \tag{4.10}$$

Since $rank[S] \neq rank[S|y_i(r)]$ $i = 1, 2$ *then the matrix equations (4.10) is inconsistent. We get*

$$S^T S = P^{-1} \begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P, P = \begin{pmatrix} -\frac{1}{10} & -\frac{1}{5} & -\frac{1}{10} & -\frac{1}{5} \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{2}{5} & -\frac{2}{5} & \frac{2}{5} & -\frac{2}{5} \end{pmatrix},$$

and by Theorem 2.1, we have

$$(S^T S)^D = P^{-1} \begin{pmatrix} \frac{1}{10} & 0 & 0 & 0 \\ 0 & \frac{1}{8} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P = \begin{pmatrix} \frac{13}{50} & \frac{1}{50} & -\frac{6}{25} & \frac{1}{50} \\ \frac{1}{50} & \frac{41}{400} & \frac{1}{50} & -\frac{9}{400} \\ -\frac{6}{25} & \frac{1}{50} & \frac{13}{50} & \frac{1}{50} \\ \frac{1}{50} & -\frac{9}{400} & \frac{1}{50} & \frac{41}{400} \end{pmatrix}.$$

By Theorem 3.2 we have $X(r) = (S^T S)^D S^T Y$. *Therefore,*

$$\begin{pmatrix} \underline{x}_{11} & \underline{x}_{12} \\ \underline{x}_{21} & \underline{x}_{22} \\ -\bar{x}_{11} & -\bar{x}_{12} \\ -\bar{x}_{21} & -\bar{x}_{22} \end{pmatrix} = \begin{pmatrix} \frac{14}{5} + \frac{7}{10}r & \frac{69}{10} + \frac{3}{5}r \\ -\frac{41}{40} + \frac{31}{40}r & -\frac{49}{20} + \frac{6}{5}r \\ -\frac{37}{10} + \frac{1}{5}r & -\frac{81}{10} + \frac{3}{5}r \\ -\frac{31}{40} + \frac{41}{40}r & \frac{1}{20} + \frac{6}{5}r \end{pmatrix},$$

is a least-square solution of (4.10). We can get fuzzy least-square solution of (4.10) by Definition 2.10. Singular value decomposition of $(S^T S)$ has the following form

$$\begin{pmatrix} -\frac{1}{\sqrt{10}} & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{10}}{5} \\ -\frac{\sqrt{10}}{5} & \frac{\sqrt{2}}{2} & 0 & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{10}}{5} \\ -\frac{\sqrt{10}}{5} & -\frac{\sqrt{2}}{2} & 0 & \frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{10}} & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{10}}{5} \\ -\frac{\sqrt{10}}{5} & \frac{\sqrt{2}}{2} & 0 & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{10}}{5} \\ -\frac{\sqrt{10}}{5} & -\frac{\sqrt{2}}{2} & 0 & \frac{1}{\sqrt{10}} \end{pmatrix}^T.$$

Then

$$(S^T S)^+ = \begin{pmatrix} \frac{13}{50} & \frac{1}{50} & -\frac{6}{25} & \frac{1}{50} \\ \frac{1}{50} & \frac{41}{400} & \frac{1}{50} & -\frac{9}{400} \\ -\frac{6}{25} & \frac{1}{50} & \frac{13}{50} & \frac{1}{50} \\ \frac{1}{50} & -\frac{9}{400} & \frac{1}{50} & \frac{41}{400} \end{pmatrix}.$$

By Theorem 3.2 we have $X(r) = (S^T S)^+ S^T Y(r)$. Therefore,

$$\begin{pmatrix} \underline{x}_{11} & \underline{x}_{12} \\ \underline{x}_{21} & \underline{x}_{22} \\ -\bar{x}_{11} & -\bar{x}_{12} \\ -\bar{x}_{21} & -\bar{x}_{22} \end{pmatrix} = \begin{pmatrix} \frac{14}{5} + \frac{7}{10}r & \frac{69}{10} + \frac{3}{5}r \\ -\frac{41}{40} + \frac{31}{40}r & -\frac{49}{20} + \frac{6}{5}r \\ -\frac{37}{10} + \frac{1}{5}r & -\frac{81}{10} + \frac{3}{5}r \\ -\frac{31}{40} + \frac{41}{40}r & \frac{1}{20} + \frac{6}{5}r \end{pmatrix}.$$

In this case, $(S^T S)^+ = (S^T S)^D$, it is shown that we can find minimal solution of inconsistent matrix equation using Drazin inverse. By Theorem 2.5 we have

$$(S^T S) \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & 0 \\ \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} (S^T S) = (S^T S).$$

Then

$$(S^T S)^{(1)} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & 0 \\ \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

A least-square solution of (4.10) by Theorem 3.3 is

$$X(r) = \begin{pmatrix} \frac{11}{4}r - \frac{3}{4} & 3r + 5 \\ -\frac{1}{4}r + \frac{3}{4} & -\frac{3}{2} \\ \frac{9}{4}r - \frac{29}{4} & 3r - 10 \\ 1 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The fuzzy least squares solutions by Definition 2.10 are given. Therefore a set of least-square solutions of inconsistent matrix equations (4.10) using generalized inverse are given.

Example 4.2. *Consider the Toeplitz matrix S with the for,*

$$S = S(s_1, s_2, \cdots, s_{n-1}) \equiv \begin{pmatrix} s_1 & s_2 & s_3 & \cdots & s_{n-1} & s_1 \\ s_{n-1} & s_1 & s_2 & \cdots & s_{n-2} & s_{n-1} \\ s_{n-2} & s_{n-1} & s_1 & \cdots & s_{n-3} & s_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s_2 & s_3 & s_4 & \cdots & s_1 & s_2 \\ s_1 & s_2 & s_3 & \cdots & s_{n-1} & s_1 \end{pmatrix}.$$

The index of matrix S is equal to 1. The matrix

$$\begin{pmatrix} -\frac{1243}{648} & -\frac{103}{648} & \frac{83}{648} & 2 \\ \frac{83}{648} & \frac{71}{648} & -\frac{139}{648} & 0 \\ -\frac{103}{648} & \frac{29}{648} & \frac{71}{648} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

is a $\{1\}$*-inverse of the matrix*

$$\begin{pmatrix} 14 & 15 & 13 & 14 \\ 13 & 19 & 22 & 13 \\ 15 & 14 & 19 & 15 \\ 14 & 15 & 13 & 14 \end{pmatrix}.$$

Therefore we get a least square solution of the inconsistent singular fuzzy matrix equations

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 3 & 1 & 2 & 3 \\ 2 & 3 & 1 & 2 \\ 1 & 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \tilde{x}_{13} & \tilde{x}_{14} \\ \tilde{x}_{12} & \tilde{x}_{22} & \tilde{x}_{23} & \tilde{x}_{24} \\ \tilde{x}_{31} & \tilde{x}_{32} & \tilde{x}_{33} & \tilde{x}_{34} \\ \tilde{x}_{41} & \tilde{x}_{42} & \tilde{x}_{43} & \tilde{x}_{44} \end{pmatrix}$$
$$= \begin{pmatrix} (2r+3, 7-2r) & (-1, -r) & (r, 2-r) & (1+r, 3-r) \\ (0, 1-r) & (0, 1-r) & (2r+1, 5-2r) & (-1+r, 1-r) \\ (r-2, -r) & (1+r, 3-r) & (-1+r, 1-r) & (1+5r, 10-4r) \\ (2r, -3r+5) & (-2, -2r) & (1+5r, 10-4r) & (2r+3, 7-2r) \end{pmatrix} \tag{4.11}$$

using $\{1\}$*-inverse. The fuzzy least-square solution of (4.11) by Definition 2.10 are given.*

5 Conclusion

A inconsistent singular fuzzy matrix equations has a set of fuzzy least-square solutions. Generalized inverses in finding minimal solution of inconsistent singular fuzzy matrix equations are applied. Solving singular fuzzy matrix equations using iterative methods is suggested.

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