Approximations of the nonlinear Painlevé transcendents

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Abstract

In this communication, we apply the new homotopy perturbation method (NHPM) to solve the first and second Painlevé transcendent. The NHPM for solving differential equations based on two component procedure and polynomial initial condition. The numerical results are compared with existing analytic results obtained by Adomian decomposition method (ADM), homotopy perturbation method (HPM), analytic continuation method and Legendre Tau method. The outcomes confirm that the scheme yields accurate and excellent results even when two components are used.

Keywords: Painlevé transcendent; new homotopy perturbation method (NHPM); differential equation; nonlinear equations.

1 Introduction

The Painlevé equations [1, 2, 3] were discovered by P. Painlevé, B. Gambier and their colleagues during studying a nonlinear second order differential equation

\[
\frac{d^2 u}{dx^2} = R(x, u, u'),
\]

where \(R(x, u, u')\) is a function rational in \(u\) and \(u'\) and analytic in \(x\). It was shown by P. Painlevé (1900) and B. Gambier (1910) that all equations of this type whose solutions do not have movable critical points (but are allowed to have fixed singular points and movable pole) can be reduced to 50 classes of equations. Moreover, 44 classes out of them are integrable by quadrature or admit reduction of order. The remaining 6 equations are irreducible; these are known as the Painlevé equations or Painlevé transcendent, and their solutions are known as the Painlevé transcendental functions. It is significant that the Painlevé equations often arise in mathematical physics. Some connections are given as follows:

The Boussinesq equation in canonical form

\[
\frac{\partial^2 w}{\partial t^2} + \frac{\partial}{\partial x} \left( w \frac{\partial w}{\partial x} \right) + \frac{\partial^4 w}{\partial x^4} = 0.
\]

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This equation arises in several physical applications: propagation of long waves in shallow water, one dimensional nonlinear lattice-waves, vibrations in a nonlinear string, and ion sound waves in plasma. The connection of Boussinesq equation with first Painlevé equation is due to Clarkson and Kruskal [4]. Assuming that a solution of Eq. (1.1) may be written in the form

\[ w(x,t) = (a_0 + a_1 t)^2 U(z) - \left( \frac{a_1 x + b_1}{a_1 t + b_0} \right)^2, \]

where \( z = x(a_1 t + a_0) + b_1 t + b_0, \) and \( a_0, a_1, b_0 \) and \( b_1 \) are arbitrary constants. The function \( U(z) \) is determined by second order differential equation

\[ U''(z) + \frac{1}{2} U^2 = c_1 z + c_2. \]

If \( c_1 \neq 0 \) the equation is reduced to the first Painlevé equation.

The Korteweg-de Vries equation in canonical form

\[ \frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial t^3} - 6 w \frac{\partial w}{\partial x} = 0. \] (1.2)

It is used in many sections of nonlinear mechanics and theoretical physics for describing one-dimensional nonlinear dispersive non-dissipative waves. Also, the mathematical modeling of moderate-amplitude shallow water surface waves is based on this equation. Assuming that a solution of Eq. (1.2) may be written in the form

\[ w(x,t) = [3(t - t_0)]^{-2/3} U(z), \]

where \( z = [3(t - t_0)]^{-1/3} (x - x_0) \), the function \( U(z) \) is determined by third order ordinary differential equation

\[ U'''(z) - z U' - 2 U - 6 U'' U = 0. \] (1.3)

A solution of Eq. (1.3) can be represented as

\[ U(z) = g'(z) + g^2(z), \]

where the function \( g(z) \) is any solution of the second Painlevé equation

\[ g''(z) - 2g^3 - zg = A, \]

where \( A \) is an arbitrary constant. In this paper, NHPM is applied to solve the nonlinear first and second Painlevé equations with Cauchy-Dirchelet condition \( y|_{x=0} = A, y'|_{x=0} = B \). Unlike the ADM [5, 6], the NHPM is free from the need to use Adomian polynomials. In this method we do not need the Lagrange multiplier, correction functional, stationary conditions, and calculating integrals, which eliminate the complications that exist in the VIM [7]. In contrast to the HPM [8, 9, 10, 11, 12], in this method, it is not required to solve the functional equations in each iteration. HPM is an especial case of HAM (Homotopy Analysis Method) and the comparison of HAM and HPM can be find in [13, 14, 15].

2 Analysis of new homotopy perturbation method

Let us consider the nonlinear differential equation

\[ \mathcal{A}(u) = f(z), \quad z \in \Omega, \] (2.4)

where \( \mathcal{A} \) is operator, \( f \) is a known function and \( u \) is a sought function. Assume that operator \( \mathcal{A} \) can be written as:

\[ \mathcal{A}(u) = \mathcal{L}(u) + \mathcal{N}(u), \]

where \( \mathcal{L}(u) \) is the linear operator and \( \mathcal{N}(u) \) is the nonlinear operator. Hence, Eq. (2.4) can be rewritten as follows:

\[ \mathcal{L}(u) + \mathcal{N}(u) = f(z), \quad z \in \Omega. \]
We define an operator \( \mathcal{H}(u) \) as:

\[
\mathcal{H}(v; p) = (1 - p)(L(v) - L(u_0)) + p(\mathcal{A}(v) - f),
\]

where \( p \in [0, 1] \) is an embedding or homotopy parameter, and \( v(z; p) : \Omega \times [0, 1] \to \mathbb{R} \) is an initial approximation of solution of the problem in Eq. (2.4). Eq. (2.5) can be written as:

\[
\mathcal{H}(v; p) = L(v) - L(u_0) + pL(u_0) + p(\mathcal{A}(v) - f(z)) = 0.
\]

Clearly, the operator equations \( \mathcal{H}(v, 0) = 0 \) and \( \mathcal{H}(v, 1) = 0 \) are equivalent to the equations \( L(v) - L(u_0) = 0 \) and \( \mathcal{A}(v) - f(z) = 0 \), respectively. Thus, a monotonous change of parameter \( p \) from zero to one corresponds to a continuous change of the trivial problem \( L(v) - L(u_0) = 0 \) to the original problem. Operator \( \mathcal{H}(v; p) \) is called a homotopy map. Next, we assume that the solution \( \mathcal{H}(v; p) \) can be written as a power series in embedding parameter \( p \), as follows:

\[
v = v_0 + pv_1.
\]

Now let us write the Eq. (2.6) in the following form

\[
L(v) = u_0(z) + p(f - \mathcal{A}(v) - u_0(z)).
\]

By applying the inverse operator \( L^{-1} \) to both sides of the Eq. (2.4), we have

\[
v = L^{-1}u_0(z) + p(L^{-1}f - L^{-1}\mathcal{A}(v) - L^{-1}u_0(z)).
\]

Suppose that the initial approximation of Eq. (2.4) has the form

\[
u_0(z) = \sum_{n=0}^{\infty} a_nP_n(z),
\]

where \( a_n, n = 0, 1, 2, \ldots \) are unknown coefficients and \( P_n(z), n = 0, 1, 2, \ldots \) are specific functions on the problem. By substituting Eqs. (2.7) and (2.9) into the Eq. (2.8), we get

\[
v_0 + pv_1 = L^{-1}\left(\sum_{n=0}^{\infty} a_nP_n(z)\right) + p\left(L^{-1}f - L^{-1}\mathcal{A}(v) - L^{-1}\left(\sum_{n=0}^{\infty} a_nP_n(z)\right)\right).
\]

Equating the coefficients of like powers of \( p \), we get following set of equations:

Coefficient of \( p^0 : v_0 = L^{-1}(\sum_{n=0}^{\infty} a_nP_n(z)). \)

Coefficient of \( p^1 : v_1 = L^{-1}f - L^{-1}(\sum_{n=0}^{\infty} v_nP_n) - L^{-1}\mathcal{A}(v_0). \)

Now, we solve these equations in such a way that \( v_1(z) = 0 \). Therefore, the approximate solution may be obtained as

\[
y_1(z) = y_{1,0}(z) = L^{-1}\left(\sum_{n=0}^{\infty} a_nP_n(z)\right).
\]

3 Approximations of the nonlinear Painlevé transcendent

3.1 Solution of the first Painlevé transcendent

The first Painlevé equation has the form

\[
\frac{d^2 u}{dx^2} = 6u^2 + x,
\]

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with Cauchy data \( y|_{x=0} = A, \ y'|_{x=0} = B. \)

To obtain the solution of Eq. (3.10) by NHPM, we construct the following homotopy:

\[
(1 - p) \left( \frac{d^2 U(x)}{dx^2} - u_0(x) \right) + p \left( \frac{d^2 U(x)}{dx^2} - u_0(x) - \left( 6U^2(x) + x \right) \right) = 0. \tag{3.11}
\]

Applying the inverse operator \( \mathcal{L}^{-1}(\bullet) = \int_0^x \int_0^\tau (\bullet) d\xi d\tau \) to the both sides of the above Eq. (3.11), we obtain

\[
U(x) = U(0) + xU'(0) + \int_0^x \int_0^\tau (U_0(\xi)) d\xi d\tau - \int_0^x \int_0^\tau \left( U_0(\xi) \left( 6U^2(\xi) + \xi \right) \right) d\xi d\tau. \tag{3.12}
\]

Suppose the solution of Eq. (3.12) to have the following form

\[
U(x) = U_0(x) + pU_1(x). \tag{3.13}
\]

Substituting Eq. (3.13) in Eq. (3.12) and equating the coefficients of like powers of \( p \), we get following set of equations

\[
U_0(x) = U(0) + xU'(0) + \int_0^x \int_0^\tau U_0(\xi) d\xi d\tau,
\]

\[
U_1(x) = \int_0^x \int_0^\tau \left( (-U_0(\xi) + 6U^2(\xi) + \xi) \right) d\xi d\tau.
\]

Assuming \( u_0(x) = \sum_{n=0}^{10} a_n P_n, \ P_k = x^k, \ U_0 = u_0 \) and solving the above equation for \( U_1(\tau) \) leads to the result

\[
U_1(x) = \left( 3A^2 - \frac{a_0}{2} \right) x^2 + \left( \frac{1 - a_0}{6} + 2AB \right) x^3 + \left( \frac{Aa_0}{2} - \frac{a_2}{12} + \frac{B^2}{2} \right) x^4 + \cdots.
\]

Vanishing \( U_1(\tau) \) lets the coefficients \( a_i, \ i = 1, 2, 3, \cdots \), the following values

\[
a_0 = 6A^2, a_1 = 1 + 12AB, a_2 = 6(6A^3 + B^3), a_3 = 2(A + 30A^2B), \cdots
\]

Therefore, we obtain the solution of Eq. (3.10) as

\[
u(x) = A + Bx + 3A^2x^2 + \frac{1}{6} (1 + 12AB)x^3 + \frac{1}{2} (6A^3 + B^3)x^4 + \frac{1}{10} (A + 30A^2B)x^5 + \cdots.
\]

### 3.2 Solution of the second Painlevé transcendent

The second Painlevé equation [16] has the form

\[
\frac{d^2 u}{dx^2} = 2u^3(x) + xu(x) + \lambda, \tag{3.14}
\]

with Cauchy data \( y|_{x=0} = A, \ y'|_{x=0} = B. \)

To obtain the solution of Eq. (3.14) by NHPM, we construct the following homotopy:

\[
(1 - p) \left( \frac{d^2 u}{dx^2} - u_0(x) \right) + p \left( \frac{d^2 u}{dx^2} - (2U^3(x) + xu(x) + \lambda) \right) = 0. \tag{3.15}
\]

Applying the inverse operator, \( \mathcal{L}^{-1}(\bullet) = \int_0^x \int_0^\tau (\bullet) d\xi d\tau \) to the both sides of the above Eq. (3.15), we obtain

\[
U(x) = U(0) + xU'(0) + \int_0^x \int_0^\tau (u_0(\xi)) d\xi d\tau - p \int_0^x \int_0^\tau \left( u_0(\xi) - (2U^3(\xi) + \xi U_0(\xi) + \lambda) \right) d\xi d\tau. \tag{3.16}
\]
Suppose the solution of Eq. (3.16) to have the following form
\[ U(x) = U(0) + pU_1(x). \quad (3.17) \]
Substituting Eq. (3.17) in the Eq. (3.16) and equating the coefficients of like powers of \( p \), we get following set of equations
\[ U_0(x) = U(0) + xU'(0) + \int_0^x \int_0^\tau u_0(\xi)d\xi d\tau, \]
\[ U_1(x) = \int_0^x \int_0^\tau (-u_0(\xi) + (2U_0(\xi) + \xi U_0(\xi) + \lambda))d\xi d\tau. \]
Assuming \( u_0(t) = \sum_{n=0}^{30} a_n t^n, P_k = r^k, U_0 = u_0 \) and solving the above equation for \( U_1(t) \) leads to the result
\[ U_1(x) = \left(3A^2 - \frac{a_0}{2}\right)x^2 + \left(1 - \frac{a_0}{6} + 2AB\right)x^3 + \left(\frac{Aa_0}{2} - \frac{a_2}{12} + \frac{B^2}{2}\right)x^4 + \cdots. \]
Vanishing \( U_1(t) \) lets the coefficients \( a_i, i = 1, 2, 3, \cdots \), the following values
\[ a_0 = 6A^2, a_1 = 1 + 12AB, a_2 = 6(6A^3 + B^3), a + 3 = 2(A + 30A^2B), \cdots \]
Therefore, we obtain the solution of the Eq. (3.10) as
\[ u(x) = A + Bx + 3A^2x^2 + \frac{1}{6}(1 + 12AB)x^3 + \frac{1}{2}(6A^3 + B^3)x^4 + \frac{1}{10}(A + 30A^2B)x^5 + \cdots. \]

4 Concluding remarks

New homotopy perturbation solutions [17] have been presented for the first and second Painlevé transcendent. The results show a very good agreement with the other methods. The major advantage of the NHPM is that it is simple, straightforward and systematic when compared with other techniques. To avoid the cumbersome of the computational work that usually arises from the traditional strategies; we present alternative numerical strategies that proved to be effective, reliable and easy to use. In HPM and ADM we reach to a set of recurrent differential equations, which must be solved consecutively to give the approximate solution of the problem. The computations associated with the solutions of first and second Painlevé transcendent in this work were performed using MATHEMATICA 7.

Acknowledgements

The author Najeeb Alam Khan is highly thankful and grateful to the Dean of Faculty of Sciences, University of Karachi, Karachi-75270, Pakistan for facilitating this research work, and grateful to the anonymous referees for their comments which substantially improved the quality of this paper.

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