On the Positive Definite Solutions of the Nonlinear Matrix Equation

$X - A^*X^{-s}A - B^*X^{-t}B = I$

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Abstract

In the present paper, the positive definite solutions for the nonlinear matrix equation

$X - A^*X^{-s}A - B^*X^{-t}B = I$, where $s, t \in (0, 1]$ are studied. The convergence of the proposed iterative methods are discussed. The necessary and sufficient conditions of the existence of positive definite solutions are derived. Finally, the results are illustrated by some numerical examples.

Keywords: Nonlinear matrix equation; Iterative methods; Positive definite solution; Property; Existence

1 Introduction

In this paper, we consider the nonlinear matrix equation:

$X - A^*X^{-s}A - B^*X^{-t}B = I$  \hspace{1cm} (1.1)

where $I$ is the identity matrix, $A, B$ are $m \times m$ nonsingular matrices, $s, t \in (0, 1]$, $X$ is unknown matrix, and all matrices are defined over the complex field.

The nonlinear matrix equation Eq. (1.1), when $s = 1$ and $t = 0$, arises in the analysis of a stationary Gaussian reciprocal processes over a finite interval [2], which discussed in many papers [1, 2, 3, 7, 8, 9, 10, 11, 12]. The positive definite solutions of Eq. (1.1) have been studied in some special cases [1, 2, 3, 7, 8, 9, 10, 11, 12]. The properties, efficient

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iterative algorithms, and perturbation estimates of the positive definite solutions of Eq. (1.1) has been investigated in some special cases [1, 2, 3, 7, 9, 10, 11, 12]. Liu and Gao [12] study the equation

$$X^s - A^T X^{-t} A = I$$

and presented the properties of the symmetric positive definite solutions of this equation.

In this paper, we investigate the properties of the positive definite solutions of Eq. (1.1). The paper is organized as follows: In Section 2, we show that Eq. (1.1) always has positive definite solutions. Also, we discuss some properties and the conditions of the existence the solutions. In Section 3, we introduce a iteration algorithm to obtain the solutions and its convergence. Finally, we give three numerical examples in Section 4 to illustrate our theoretical results.

The following notations are used throughout this paper. The notation $A \geq 0$ ($A > 0$) means that $A$ is positive semidefinite (positive definite), $A^*$ denotes the complex conjugate transpose of $A$, and $I$ is the identity matrix. Moreover, $A \geq B$ ($A > B$) is used as a different notation for $A - B \geq 0$ ($A - B > 0$). We denote by $\rho(A)$ the spectral radius of $A$, by $\sigma_1(A)$ and $\sigma_m(A)$ the maximal and minimal singular values of $A$, respectively. The norm used in this paper is the spectral norm of the matrix $A$, i.e. $\|A\| = \sqrt{\rho(A^*A)}$ unless otherwise noted.

2 Some Properties of the Solutions

In this section, we will discuss some properties of Eq. (1.1) and obtain the conditions for the existence and properties of the solutions of Eq. (1.1).

**Lemma 2.1.** [4, 5], If $P > Q > 0$ (or $P \geq Q > 0$), then $P^\alpha > Q^\alpha$ (or $P^\alpha \geq Q^\alpha > 0$) for all $\alpha \in (0, 1]$, and $P^\alpha < Q^\alpha$ (or $0 < P^\alpha \leq Q^\alpha$) for all $\alpha \in [-1, 0)$.

**Theorem 2.1.** If $X$ is a positive definite solution of Eq. (1.1), then

$$I < X < I + A^* A + B^* B$$  \hspace{1cm} (2.2)

**Proof.** The left inequality is evident. To prove the right inequality, from the inequality $X > I$ and lemma 2.1 we get

$$A^* X^{-s} A < A^* A, \quad B^* X^{-t} B < B^* B.$$ 

Therefore

$$X = I + A^* X^{-s} A + B^* X^{-t} B < I + A^* A + B^* B,$$

i.e.

$$I < X < I + A^* A + B^* B.$$ \hfill \square

**Theorem 2.2.** Eq. (1.1) has a positive definite solution if and only if the matrices $A, B$ can factor as

$$A = (W^* W)^{t/2} Z_1, \quad B = (W^* W)^{t/2} Z_2$$  \hspace{1cm} (2.3)

where $W$ is a nonsingular matrix and \begin{pmatrix} W^{-1} \\ Z_1 W^{-1} \\ Z_2 W^{-1} \end{pmatrix} is column-orthonormal.
Proof. Let Eq. (1.1) have a positive definite solution \( X \), then \( X = W^*W \), where \( W \) is a nonsingular matrix. Then, Eq. (1.1) can be rewritten as

\[
W^*W - A^*(W^*W)^{-s/2}(W^*W)^{-s/2}A - B^*(W^*W)^{-t/2}(W^*W)^{-t/2}B = I
\]

(2.4)

Let \( Z_1 = (W^*W)^{-s/2}A \), \( Z_2 = (W^*W)^{-t/2}B \), then \( A = (W^*W)^{s/2}Z_1 \), \( B = (W^*W)^{t/2}Z_2 \), and the Eq. (2.4) equivalently the equation

\[
W^*W - Z_1^tZ_1 - Z_2^tZ_2 = I
\]

(2.5)

Eq. (2.5) means that the column \( \begin{bmatrix} W^{-1} \\ Z_1W^{-1} \\ Z_2W^{-1} \end{bmatrix} \) is orthonormal.

Conversely, if \( A, B \) have the factorization (2.3) and satisfying Eq. (2.5), Let \( X = W^*W \), then \( X \) a positive definite matrix , and we have

\[
X - A^*X^{-s}A - B^*X^{-t}B = W^*W - Z_1^t(W^*W)^{s/2}(W^*W)^{-s}(W^*W)^{s/2}Z_1
- Z_2^t(W^*W)^{t/2}(W^*W)^{-t}(W^*W)^{t/2}Z_2
= W^*W - Z_1^tZ_1 - Z_2^tZ_2
= I.
\]

Hence \( X \) is a positive definite solution of Eq. (1.1).

\[\square\]

3 The Algorithm and its Convergence

In this section, we consider the fixed point iteration algorithm. The following iterative process to calculate the positive definite solution of Eq. (1.1)

\[
X_0 = I, \quad X_{k+1} = I + A^*X_k^{-s}A + B^*X_k^{-t}B, \quad k = 0, 1, 2, ...
\]

(3.6)

**Theorem 3.1.** If there is a real number \( \alpha \), such that \( 1 < \alpha < 2 \) and

\[
A^*A, \quad B^*B < \frac{(\alpha - 1)}{2}I
\]

then the Eq.(1.1) has a positive definite solution.

**Proof.** We consider the sequence (3.6). For \( X_1 \) we have

\[
X_1 = I + A^*A + B^*B > I = X_0
\]

i.e. \( X_0 < X_1 \). By Using the conditions \( A^*A, \quad B^*B < \frac{(\alpha - 1)}{2}I \) we have

\[
X_1 = I + A^*A + B^*B < \left[ 1 + \frac{(\alpha - 1)}{2} + \frac{(\alpha - 1)}{2} \right] I = \alpha I.
\]

From \( X_1 < \alpha I \) and Lemma 2.1

\[
X_2 = I + A^*X_1^{-s}A + B^*X_1^{-t}B
> I + (\alpha)^{-s}A^*A + (\alpha)^{-t}B^*B
> I.
\]
Consequently $X_0 < X_2$. Using the relation $X_0 < X_1$ and Lemma 2.1, we obtain $X_0^{-s} > X_1^{-s}$, $X_0^{-t} > X_1^{-t}$, and hence

$$X_2 = I + A^*X_1^{-s}A + B^*X_1^{-t}B$$

$$< I + A^*X_0^{-s}A + B^*X_0^{-t}B$$

$$= X_1$$

i.e. $X_0 < X_2 < X_1$.

By the same way we can prove that $X_3 < X_1$ and $X_2 < X_3$, consequently $X_0 < X_2 < X_3 < X_1$.

We receive by analogy that for each two integers $k, r$ the following is true

$$X_0 \leq X_{2k} < X_{2k+2} < X_{2r+3} < X_{2r+1} \leq X_1.$$  

We observe that the two subsequences $\{X_{2k}\}, \{X_{2r+1}\}$ have the same boundaries. To prove that these two subsequences converge to positive definite matrix, we have

$$\|X_{2k+1} - X_{2k}\| = \|A^*(X_{2k}^{-s} - X_{2k-1}^{-s})A + B^*(X_{2k}^{-t} - X_{2k-1}^{-t})B\|$$

$$\leq \|A\|^2\|X_{2k}^{-s} - X_{2k-1}^{-s}\| + \|B\|^2\|X_{2k}^{-t} - X_{2k-1}^{-t}\|$$

$$= \|A\|^2\|X_{2k}^{-s}(X_{2k}^{-s} - X_{2k}^{-s})X_{2k}^{-s}\| + \|B\|^2\|X_{2k}^{-t}(X_{2k}^{-t} - X_{2k}^{-t})X_{2k}^{-t}\|$$

$$\leq \|A\|^2\|X_{2k}^{-s}\|\|X_{2k}^{-s}\|\|X_{2k}^{-s}\|\|X_{2k}^{-t}\|\|X_{2k}^{-t}\|\|X_{2k}^{-t}\|$$

Since $X_{k} > X_0 = I$, $\forall k = 1, 2, 3, \ldots$, then by using Lemma 2.1 we have $X_{k}^{-s}, X_{k}^{-t} < I$, $\forall k = 1, 2, 3, \ldots$.

Consequently

$$\|X_{2k+1} - X_{2k}\| \leq \|A\|^2\|X_{2k-1}^{-s} - X_{2k}^{-s}\| + \|B\|^2\|X_{2k-1}^{-t} - X_{2k}^{-t}\|$$

(3.7)

Since $X_{2k-1}^{-s} > X_{2k}^{-s}$, $\forall k = 1, 2, 3, \ldots$, then $P = X_{2k-1}^{-s} - X_{2k}^{-s}$ is a positive definite solution of the matrix equation

$$X_{2k-1}^{-s}P + PX_{2k}^{-s} = X_{2k-1}^{2s} - X_{2k}^{2s}.$$  

(3.8)

According to Theorem (3) in [10], then the solution of (3.8) is

$$P = \int_{0}^{\infty} e^{-X_{2k-1}^{-s}t} (X_{2k-1}^{2s} - X_{2k}^{2s}) e^{-X_{2k}^{-t}t} dt.$$  

Hance

$$\|X_{2k-1}^{-s} - X_{2k}^{-s}\| \leq \|X_{2k-1}^{2s} - X_{2k}^{2s}\| \int_{0}^{\infty} \|e^{-X_{2k-1}^{-s}t}\|\|e^{-X_{2k}^{-t}t}\| dt$$

$$\leq \|X_{2k-1}^{2s} - X_{2k}^{2s}\| \int_{0}^{\infty} \|e^{-t}\|\|e^{-t}\| dt$$

$$= \|X_{2k-1}^{2s} - X_{2k}^{2s}\| \int_{0}^{\infty} e^{-2t} dt$$

$$= \frac{1}{2}\|X_{2k-1}^{2s} - X_{2k}^{2s}\|$$

$$\leq \frac{1}{2}\|X_{2k-1}^{2s} - X_{2k}^{2s}\|.$$
After $n$ times as above, we get
\[
\|X_{2k-1}^{s} - X_{2k}^{s}\| \leq \frac{1}{2^n} \|X_{2k-1}^{2ns} - X_{2k}^{2ns}\|. \tag{3.9}
\]
Similarly
\[
\|X_{2k-1}^{t} - X_{2k}^{t}\| \leq \frac{1}{2^n} \|X_{2k-1}^{2nt} - X_{2k}^{2nt}\|. \tag{3.10}
\]
By using (3.9) and (3.10) in (3.7), we have
\[
\|X_{2k+1} - X_{2k}\| \leq \frac{1}{2^n} \|A\| \|X_{2k-1}^{2ns} - X_{2k}^{2ns}\| + \frac{1}{2^n} \|B\| \|X_{2k-1}^{2nt} - X_{2k}^{2nt}\|. \tag{3.11}
\]
By using the conditions $A, B < \frac{(\alpha - 1)}{2} I$, we have
\[
\|X_{2k+1} - X_{2k}\| \leq \frac{(\alpha - 1)}{2^{n+1}} \left( \|X_{2k-1}^{2ns} - X_{2k}^{2ns}\| + \|X_{2k-1}^{2nt} - X_{2k}^{2nt}\| \right), \tag{3.11}
\]
i.e.
\[
\|X_{2k+1} - X_{2k}\| \to \infty, \quad \text{as} \ n \to \infty
\]
This proves the theorem. \hfill \Box

From the given proof of the above theorem, we can deduce the following corollary.

**Corollary 3.1.** From inequality (3.11), we have the following upper bound
\[
\max(\|X_{2k+1} - X\|, \|X - X_{2k}\|) \leq \frac{(\alpha - 1)}{2^{n+1}} \left( \|X_{2k-1}^{2ns} - X_{2k}^{2ns}\| + \|X_{2k-1}^{2nt} - X_{2k}^{2nt}\| \right) \tag{3.12}
\]

**4 Numerical Experiments**

We made numerical experiments for computing of a positive definite solution of the Eq. (1.1). The solution is computed for some different matrices $A, B$ with different orders. Denote $X$ the solution which is obtained by the iterative method (3.6) and $\epsilon_1(X) = \|X - X_k\|, \epsilon_2(X) = \|X_k - A^sX_k^{-s}A - B^sX_k^{-t}B - I\|$. For computing $X^\alpha \forall \alpha \in (0, 1]$ we use the iterative process:
\[
Z_0X = XZ_0, \quad Z_{k+1} = \alpha \left[ \frac{1}{\alpha} - 1 \right] Z_k + Z_k^{1-\frac{1}{\alpha}} X, \quad k = 0, 1, 2, \ldots \tag{4.13}
\]
See[6].

**Example 4.1.** Consider Eq. (1.1) with
\[
A = \begin{cases} 
  a_{ij} = \frac{i}{12m^2} & \text{if} \ i = j \\
  a_{ij} = \frac{i}{14+4ms^2} & \text{if} \ i < j \\
  a_{ij} = 0.1j & \text{if} \ i > j 
\end{cases}, \quad B = \begin{cases} 
  b_{ij} = \frac{i}{m} & \text{if} \ i = j \\
  b_{ij} = \frac{i+j}{nm} & \text{if} \ i \neq j 
\end{cases}
\]
Let $m = 5$, $s = \frac{1}{4}$, $t = \frac{1}{5}$, by computation, we get

$$X = \begin{pmatrix}
1.15536 & 0.179941 & 0.190267 & 0.178352 & 0.140615 \\
0.179941 & 1.27419 & 0.259518 & 0.234098 & 0.164595 \\
0.190267 & 0.259518 & 1.38375 & 0.280102 & 0.177661 \\
0.178352 & 0.234098 & 0.280102 & 1.42338 & 0.175626 \\
0.140615 & 0.164595 & 0.177661 & 0.175626 & 1.32401
\end{pmatrix}$$

$$\lambda_i(X) = \{2.1351, 1.19833, 1.1353, 1.06866, 1.02329\}$$

The numerical results are given in the following table.

**Table 1**

<table>
<thead>
<tr>
<th>k</th>
<th>$\epsilon_1(X)$</th>
<th>$\epsilon_2(X)$</th>
</tr>
</thead>
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<td>4.66935e-01</td>
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<td>8</td>
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<td>8.44072e-05</td>
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<td>10</td>
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<td>8.73761e-06</td>
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<tr>
<td>12</td>
<td>8.49267e-07</td>
<td>1.13965e-06</td>
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<tr>
<td>14</td>
<td>8.55280e-08</td>
<td>1.13708e-07</td>
</tr>
<tr>
<td>15</td>
<td>2.81797e-08</td>
<td>2.77357e-08</td>
</tr>
</tbody>
</table>

**Example 4.2.** Consider Eq. (1.1) with

$$A = 0.15 \begin{pmatrix} 1 & 0 & 4 \\ 1 & 2 & 1 \\ 1 & 2 & 0 \end{pmatrix}, \quad B = 0.02 \begin{pmatrix} 5 & 2 & 4 \\ 6 & 4 & 6 \\ 3 & 1 & 4 \end{pmatrix}, \quad s = \frac{1}{2}, \quad t = \frac{1}{5}$$

By computation, we get

$$X = \begin{pmatrix} 1.08413 & 0.0894612 & 0.125084 \\ 0.0894612 & 1.16621 & 0.0401065 \\ 0.125084 & 0.0401065 & 1.38788 \end{pmatrix}$$

$$\lambda_i(X) = \{1.44977, 1.18259, 1.00586\}$$

The numerical results are given in the following table.

**Table 2**

<table>
<thead>
<tr>
<th>k</th>
<th>$\epsilon_1(X)$</th>
<th>$\epsilon_2(X)$</th>
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<td>13</td>
<td>8.14788e-08</td>
<td>3.82555e-08</td>
</tr>
</tbody>
</table>
Example 4.3. Consider Eq. (1.1) with 

\[
A = 0.02 \begin{pmatrix} 1 & 2 & 4 & 4 \\ 8 & 3 & 2 & 1 \\ 6 & 1 & 5 & 4 \\ 6 & 2 & 1 & 5 \end{pmatrix}, \quad B = 0.04 \begin{pmatrix} 4 & 5 & 2 & 1 \\ 2 & 4 & 2 & 6 \\ 3 & 1 & 2 & 4 \\ 3 & 1 & 1 & 2 \end{pmatrix}, \quad s = \frac{1}{8}, \quad t = \frac{1}{3}
\]

By computation, we get 

\[
X = \begin{pmatrix} 1.11005 & 0.0673803 & 0.052928 & 0.0757671 \\ 0.0673803 & 1.07172 & 0.0393886 & 0.0616113 \\ 0.052928 & 0.0393886 & 1.03745 & 0.0527413 \\ 0.0757671 & 0.0616113 & 0.0527413 & 1.10948 \end{pmatrix}
\]

\[
\lambda_i(X) = \{1.26719, 1.03487, 1.02042, 1.00621\}
\]

The numerical results are given in the following table.

<table>
<thead>
<tr>
<th>k</th>
<th>(e_1(X))</th>
<th>(e_2(X))</th>
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</table>

5 Conclusion

In this paper we consider a nonlinear matrix equation Eq. (1.1) where \(s, t \in (0, 1]\). We considered a recursion algorithms for solving matrix equation from which a positive definite solution can be calculated. We derived the necessary and sufficient conditions for existence of the positive definite solutions for equation.

References


