The Use of Iterative Methods to Solve
Two-Dimensional Nonlinear Volterra-Fredholm
Integro-Differential Equations

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Abstract
In this present paper, we solve a two-dimensional nonlinear Volterra-Fredholm integro-differential equation by using the following powerful, efficient but simple methods:
(i) Modified Adomian decomposition method (MADM),
(ii) Variational iteration method (VIM),
(iii) Homotopy analysis method (HAM) and
(iv) Modified homotopy perturbation method (MHPM).
The uniqueness of the solution and the convergence of the proposed methods are proved in detail. Numerical examples are studied to demonstrate the accuracy of the presented methods.

Keywords: Two-dimensional Volterra and Fredholm integral equations; Integro-differential equations; Modified Adomian decomposition method; Variational iteration method; Homotopy analysis method; Modified homotopy perturbation method.

1 Introduction
Generally, real-world physical problems are modelled as differential, integral and integro-differential equations. Since finding the solution of these equations is too complicated, in recent years a lot of attention has been devoted by researchers to find the analytical and numerical solution of this equations. Some mathematician have been focusing on the development of more advanced and efficient methods for one-dimensional integral equations and integro-differential equations such as the Taylor polynomials method [7, 8, 40, 52],

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semiaanalytical-numerical techniques such as the Adomian decomposition method [48] and modified Adomian decomposition method [6, 14, 49]. Many other authors have studied solutions of two-dimensional nonlinear equations by using various methods, such as solving a class of two-dimensional linear and nonlinear Volterra integral equations by the differential transform method [44], extrapolation of nystrom solution for two-dimensional nonlinear Fredholm integral equations [17], Richardson extrapolation of iterated discrete Galerkin solution for two-dimensional nonlinear Fredholm integral equations [18], cubic spline-projection method for two-dimensional equations of scattering theory [13], numerical solution of two-dimensional nonlinear Fredholm integral equations of the second kind by spline functions [9], a fast numerical solution method for two-dimensional Fredholm integral equations of the second kind [51], a class of two-dimensional dual integral equations and its application [45] and new differential transformation approach for two-dimensional Volterra integral equations [26]. In this work, we employ the MADM, VIM, HAM and MHPM to solve the two-dimensional nonlinear Volterra-Fredholm integro-differential equation as follows:

\[
\sum_{j=0}^{k} p_j(x, y)u(x, y) = f(x, y) + \int_{a}^{x} \int_{\phi} k(x, y)G(u(x, y)) \, dy \, dx, \quad (x, y) \in \mathcal{J} = [a, b] \times \phi.
\]

with initial conditions

\[
u(a, y) = g_r(y), \quad r = 0, 1, \ldots, k - 1, \quad a \leq y \leq x, \quad a \leq x \leq b, \quad \phi = [a, b].
\]

where \( a, b \) are constant values, \( u(x, y) \) is an unknown function, the functions \( f(x, y), k(x, y), \) and \( G(u(x, y)), l \geq 0 \) are analytic functions on \( \mathcal{J} \) and functions \( p_j(x, y), j = 0, 1, \ldots, k \), \( P_k(x, y) \neq 0 \) are given. The organization of this paper is as follows: In section 2, the iterative methods MADM, VIM, HAM and MHPM are introduced for solving Eq. (1.1). Also, the existence and uniqueness of the solution and convergence of the mentioned proposed methods are brought in section 3. Finally, in section 4, two numerical examples are presented to illustrate the accuracy of these methods. A brief conclusion is given in section 5.

We can write Eq. (1.1) as follows:

\[
\begin{align*}
L^{-1}(\cdot) & = \int_{a}^{x} \int_{a}^{x} \ldots \int_{a}^{x} (\cdot) \, dx \, dx \ldots \, dx.
\end{align*}
\]

we can obtain the term \( g_0(y) + \sum_{r=0}^{k-2} \int_{a}^{x} \frac{1}{(r)!} (x - y)^r g_{r+1}(y) \, dy, \) from the initial conditions. From [48], we have

\[
L^{-1}(\int_{a}^{x} \int_{\phi} k(x, y)G(u(x, y)) \, dy \, dx) = \int_{a}^{x} \int_{\phi} (x - y)^k k(x, y)G(u(x, y)) \, dy \, dx.
\]
Consider the general equation:

\[ F u = g, \]

Let us first recall the basic principles of the Adomian decomposition methods \[11, 53, 42\].

2.1 The methods

In Eq. (1.3), we assume

\[ \sum_{j=0}^{k-1} L^{-1}\left( \frac{p_j(x, y)}{p_k(x, y)} \right) u^{(j)}_x(x, y) = \sum_{j=0}^{k-1} \int_a^x \frac{(x-y)^{k-1}}{(k-1)!} \frac{p_j(x, y)}{p_k(x, y)} u^{(j)}(x, y) \ dy, \]  

(1.5)

By substituting Eq. (1.4) and Eq. (1.5) in Eq. (1.3) we obtain

\[ u(x, y) = L^{-1}\left( \frac{f(x, y)}{p_k(x, y)} \right) + g_0(y) + \sum_{r=0}^{k-2} \frac{k}{(r)!} (x-y)^r g_{r+1}(y) \ dy \]

\[ + \int_a^x \int_\phi \frac{(x-y)^k k(x,y) G(u^{(t)}(x,y))}{p_k(x,y)} \ dx \ dy - \sum_{j=0}^{k-1} \int_a^x \frac{(x-y)^{k-1}}{(k-1)!} \frac{p_j(x, y)}{p_k(x, y)} u^{(j)}(x, y) \ dy. \]  

(1.6)

We set,

\[ L^{-1}\left( \frac{f(x, y)}{p_k(x, y)} \right) + g_0(y) + \sum_{r=0}^{k-2} \frac{k}{(r)!} (x-y)^r g_{r+1}(y) \ dy = F_1(x, y), \]

\[ k_1(x, y) = \int_a^x \frac{(x-y)^k k(x,y) G(u^{(t)}(x,y))}{p_k(x,y)} \ dx, \]

\[ k_2(x, y) = \frac{(x-y)^{k-1}}{(k-1)!} \frac{p_j(x, y)}{p_k(x, y)}. \]

So, we have one-dimensional nonlinear integro-differential equation as follows:

\[ u(x, y) = F_1(x, y) + \int_a^x k_1(x, y) G(u^{(t)}(x,y)) \ dy - \sum_{j=0}^{k-1} \int_a^x k_2(x, y) u^{(j)}_x(x, y) \ dy. \]  

(1.7)

In Eq. (1.7), we assume \( F_1(x, y) \) is bounded for all \( x, y \) in \( \phi \) and

\[ |k_1(y,t)| \leq N_1, \]

\[ |k_2(y,t)| \leq N_1, \quad j = 0, 1, \ldots, k-1, \forall y, t \in J'. \]

Also, we suppose the nonlinear terms \( G(u(x,y)) \) and \( D^j(u(x,y)) \) are Lipschitz continuous with

\[ |G(u(x,y)) - G(u^*(x,y))| \leq d |u(x,y) - u^*(x,y)|, \]

\[ |D^j(u(x,y)) - D^j(u^*(x,y))| \leq Z_j |u(x,y) - u^*(x,y)|, \quad j = 0, 1, \ldots, k-1. \]

\[ \gamma = (b - a) (d N_1 + k Z N), \quad Z = \max |Z_j|, \quad N = \max |N_{1j}|, \quad j = 0, 1, \ldots, k-1. \]

2 The methods

In what follows we will highlight briefly the main point of each the methods, where details can be found in \[1, 5, 16, 29, 30, 38, 47\].

2.1 The modified Adomian decomposition method

Let us first recall the basic principles of the Adomian decomposition methods \[11, 53, 42\].

Consider the general equation: \( Fu = g \), where \( F \) represents a general nonlinear differential operator involving both linear and nonlinear terms, the linear term is decomposed into \( L + R \), where \( L \) is easily invertible and \( R \) is the remainder of the linear operator. For
convenience, L may be taken as the highest order derivate. Thus the equation may be written as
\[ L(u) + R(u) + N(u) = g(x), \]  
where \( Nu \) represents the nonlinear terms. Solving \( Lu \) from Eq.(2.8), we have
\[ L(u) = g(x) - R(u) - N(u), \]  
because \( L \) is invertible, the equivalent expression is
\[ u = f(x) - L^{-1}R(u) - L^{-1}N(u), \]  
where the function \( f(x) \) represents the term arising from integrate the source term \( g(x) \).
Therefore, \( u \) can be presented as a series
\[ u = \sum_{n=0}^{\infty} u_n, \]  
where \( u_0 \) identified as \( f(x) \), and \( u_n (n > 0) \) is to be determined. The nonlinear term \( Nu = G(u) \) will be decomposed by the infinite series of Adomian polynomials
\[ G(u) = \sum_{n=0}^{\infty} A_n, \]  
where \( A_n, n \geq 0 \) are the Adomian polynomials determined formally as follows:
\[ A_n = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} [N(\sum_{i=0}^{\infty} \lambda^i u_i)] \right]_{\lambda=0}. \]  
Adomian polynomials were introduced in [12, 14, 50] as
\[ A_0 = G(u_0), \]  
\[ A_1 = u_1 G'(u_0), \]  
\[ A_2 = u_2 G'(u_0) + \frac{1}{2!} u_1^2 G''(u_0), \]  
\[ A_3 = u_3 G'(u_0) + u_1 u_2 G''(u_0) + \frac{1}{3!} u_1^3 G'''(u_0), \ldots \]  
Now, substituting Eq. (2.11) and Eq. (2.12) into Eq. (2.10), we obtain
\[ \sum_{n=0}^{\infty} u_n = f(x) - L^{-1}R \sum_{n=0}^{\infty} u_n - L^{-1} \sum_{n=0}^{\infty} A_n \]  
consequently, we can write
\[ u_0 = f(x), \]  
\[ u_1 = -L^{-1}R(u_0) - L^{-1}(A_0), \]  
\[ u_2 = -L^{-1}R(u_1) - L^{-1}(A_1), \]  
\[ \vdots \]  
\[ u_{n+1} = -L^{-1}R(u_n) - L^{-1}(A_n), \quad n \geq 0 \]  
(2.16)
All of \( u_n \) are calculable, and 
\[ u = \sum_{n=0}^{\infty} u_n. \]
Since the series converges and does so very rapidly, the n-term partial sum \( \varphi_n = \sum_{i=0}^{n-1} u_i \) can serve as a practical solution.

The modified Adomian decomposition method (MADM) was introduced by Wazwaz in [48]. The modified forms was established based on the assumption that the function \( f(x) \) can be divided into two parts, namely \( f_1(x) \) and \( f_2(x) \). Under this assumption we set
\[
 f(x) = f_1(x) + f_2(x). \tag{2.17}
\]
Accordingly, a slight variation was proposed only on the components \( u_0 \) and \( u_1 \). The suggestion was that only the part \( f_1(x) \) be assigned to the zeroth component \( u_0 \), whereas the remaining part \( f_2(x) \) be combined with the other terms given in Eq.(2.16) to define \( u_1 \). Consequently, the modified recursive relation
\[
 u_0 = f_1(x), \\
 u_1 = f_2(x) - L^{-1}R(u_0) - L^{-1}(A_0), \\
 \vdots \\
 u_{n+1} = -L^{-1}R(u_n) - L^{-1}(A_n), \quad n \geq 1,
\tag{2.18}
\]
was developed.

2.1.1 The application of MADM

In this part, the extended the modified Adomian decomposition method [35] is used to find approximate of two-dimensional nonlinear Volterra-Fredholm integro-differential equation Eq. (1.1) , according to the MADM, we can write the iterative formula Eq. (2.18) as follows:
\[
 u_0(x, y) = f_1(x, y), \\
 u_1(x, y) = f_2(x, y) + \int_a^x k_1(x, y) A_0 \, dy - \sum_{j=0}^{k-1} \int_a^x k_2(x, y) L_{0_j} \, dy, \\
 \vdots \\
 u_{n+1}(x, y) = \int_a^x k_1(x, y) A_n \, dy - \sum_{j=0}^{k-1} \int_a^x k_2(x, y) L_{n_j} \, dy, \quad n \geq 1.
\tag{2.19}
\]
The nonlinear terms \( G(u^{(l)}(x, y)) \) and \( D^{(l)}(u(x, y)) \) \((D^{(l)} = \frac{\partial^l u(x, y)}{\partial x^l} \) is derivative operator), are usually represented by an infinite series of the so called Adomian polynomials as follows:
\[
 G(u^{(l)}(x, y)) = \sum_{i=0}^{\infty} A_i, \quad D^{(l)}(u(x, y)) = \sum_{i=0}^{\infty} L_{i_j}.
\]
where \( A_i \) and \( L_{i_j}(i \geq 0, \ j = 0, 1, \ldots, k-1) \) are the Adomian polynomials were introduced in [12].

2.2 Variational iteration method

For the purpose of illustration of the methodology to the variational iteration method , we begin by considering a nonlinear differential equation the formal form [2, 3, 4, 16]
\[
 L(u) + N(u) = g(x), \tag{2.20}
\]
where $L$ and $N$ are linear and nonlinear operators respectively and $g(t)$ is a known analytical function. He [27, 28] introduced method where a correction functional for Eq. (2.20) can be written as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)[Lu_n(\tau)) + N\tilde{u}_n(\tau)) - g(\tau)]d\tau, \quad n \geq 0,$$

(2.21)

where $\lambda$ is a general Lagrange multiplier [31], which can be identified optimally via variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$.

It is obvious now that the main steps of Hes variational iteration method require first the determination of the Lagrangian multiplier $\lambda$ that will be identified optimally. Having determined the Lagrangian multiplier, the successive approximations $u_{n+1}, n \geq 0$, of the solution $u$ will be readily obtained upon using any selective function $u_0$. Consequently, the solution

$$u(x) = \lim_{n \to \infty} u_n(x).$$

(2.22)

### 2.2.1 The application of VIM

In this part, the extended the variational iteration method is used to find approximate of two-dimensional nonlinear Volterra-Fredholm integro-differential equation, according to the VIM, we can write the iteration formula as follows:

$$u_{n+1}(x, y) = u_n(x, y) + \int_0^y \lambda(\tau)[u_n(x, \tau) - F_1(x, \tau) - \int_a^x k_1(x, \tau) G(u_n(\tau)) d\tau$$

$$- \sum_{j=0}^{k-1} \int_a^x k_2(x, \tau) (u_n)^{(j)}(x, \tau) d\tau] d\tau.$$

(2.23)

To find the optimal $\lambda$, we proceed as follows:

$$\delta u_{n+1}(x, y) = \delta u_n(x, y) + \delta \int_0^y \lambda(\tau)[u_n(x, \tau) - F_1(x, \tau)$$

$$- \int_a^x k_1(x, \tau) G(u_n(\tau)) d\tau - \sum_{j=0}^{k-1} \int_a^x k_2(x, \tau) (u_n)^{(j)}(x, \tau) d\tau] d\tau$$

$$= 0.$$

Then we apply the following stationary conditioned:

$$\lambda' = 0, \quad 1 + \lambda = 0.$$
2.3 Homotopy analysis method

Consider
\[ N[u] = 0, \]
where \( N \) is a nonlinear operator, \( u(x, y) \) is unknown function. Let \( u_0(x, y) \) denote an initial guess of the exact solution \( u(x, y) \), \( h \neq 0 \) an auxiliary parameter, \( H(x, y) \neq 0 \) an auxiliary function, and \( L \) an auxiliary linear operator with the property \( L[r(x, y)] = 0 \) when \( r(x, y) = 0 \). Then using \( q \in [0, 1] \) as an embedding parameter, we construct a homotopy as follows:

\[
(1 - q)L[\phi(x, y; q) - u_0(x, y)] - qhH(x, y)N[\phi(x, y; q)] = \hat{H}[\phi(x, y; q); u_0(x, y), H(x, y), h, q].
\]

(2.25)

It should be emphasized that we have great freedom to choose the initial guess \( u_0(x, y) \), the auxiliary linear operator \( L \), the non-zero auxiliary parameter \( h \), and the auxiliary function \( H(x, y) \). Enforcing the homotopy Eq. (2.25) to be zero, i.e.,

\[
\hat{H}[\phi(x, y; q); u_0(x, y), H(x, y), h, q] = 0,
\]

(2.26)

we have the so-called zero-order deformation equation

\[
(1 - q)L[\phi(x, y; q) - u_0(x, y)] = qhH(x, y)N[\phi(x, y; q)].
\]

(2.27)

when \( q = 0 \), the zero-order deformation Eq. (2.27) becomes

\[
\phi(x, y; 0) = u_0(x, y),
\]

(2.28)

and when \( q = 1 \), since \( h \neq 0 \) and \( H(x, y) \neq 0 \), the zero-order deformation Eq.(2.27) is equivalent to

\[
\phi(x, y; 1) = u(x, y).
\]

(2.29)

Thus, according to Eq. (2.28) and Eq. (2.29), as the embedding parameter \( q \) increases from 0 to 1, \( \phi(x, y; q) \) varies continuously from the initial approximation \( u_0(x, y) \) to the exact solution \( u(x, y) \). Such a kind of continuous variation is called deformation in homotopy [19, 20, 49].

Due to Taylor’s theorem, \( \phi(x, y; q) \) can be expanded in a power series of \( q \) as follows:

\[
\phi(x, y; q) = u_0(x, y) + \sum_{m=1}^{\infty} u_m(x, y)q^m,
\]

(2.30)

where,

\[
u_m(x, y) = \frac{1}{m!} \frac{\partial^m \phi(x, t; q)}{\partial q^m} \bigg|_{q=0}.
\]

Let the initial guess \( u_0(x, y) \), the auxiliary linear parameter \( L \), the nonzero auxiliary parameter \( h \) and the auxiliary function \( H(x, y) \) be properly chosen so that the power series Eq. (2.30) of \( \phi(x, y; q) \) converges at \( q = 1 \), then, we have under these assumptions the solution series

\[
u(x, y) = \phi(x, t; 1) = u_0(x, y) + \sum_{m=1}^{\infty} u_m(x, y).
\]

(2.31)
From Eq. (2.30), we can write Eq. (2.27) as follows:

\[(1 - q)L[\phi(x, y, q) - u_0(x, y)] = (1 - q)L[\sum_{m=1}^{\infty} u_m(x, y) q^m] = q h H(x, y)N[\phi(x, q)]\]

then,

\[L[\sum_{m=1}^{\infty} u_m(x, y) q^m] - q L[\sum_{m=1}^{\infty} u_m(x, y) q^m] = q h H(x, y)N[\phi(x, y, q)], \quad (2.32)\]

by differentiating Eq. (2.30) \(m\) times with respect to \(q\), we obtain

\[\{L[\sum_{m=1}^{\infty} u_m(x, y) q^m] - q L[\sum_{m=1}^{\infty} u_m(x, y) q^m]\}^{(m)} = \{q h H(x, y)N[\phi(x, q)]\}^{(m)} = m!L[u_m(x, y) - u_{m-1}(x)] = h H(x, y) m \frac{\partial^{m-1} N[\phi(x, y, q)]}{\partial q^{m-1}} \big|_{q=0},\]

therefore,

\[L[u_m(x, y) - \chi_m u_{m-1}(x, y)] = hH(x, y)\Re_m(u_{m-1}(x, y)), \quad (2.33)\]

where,

\[\Re_m(u_{m-1}(x, y)) = \frac{1}{(m - 1)!} \frac{\partial^{m-1} N[\phi(x, y, q)]}{\partial q^{m-1}} \big|_{q=0}, \quad (2.34)\]

and

\[\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}\]

Note that the high-order deformation Eq. (2.33) is governing the linear operator \(L\), and the term \(\Re_m(y_{m-1}(x, y))\) can be expressed simply by Eq. (2.34) for any nonlinear operator \(N\).

### 2.3.1 The application of HAM

In this part, the extended the homotopy analysis method [1] is used to find approximate of two-dimensional nonlinear Volterra-Fredholm integro-differential equation Eq. (1.1), according to the HAM, let

\[N[u(x, y)] = u(x, y) - F_1(x, y) - \int_a^x k_1(x, y) G(u^{(l)}(x, y)) \, dy + \sum_{j=0}^{k-1} \int_a^x k_2(x, y) D^j(u(x, y)) \, dy,\]

so,

\[\Re_m(u_{m-1}(x, y)) = u_{m-1}(x, y) - \int_a^x k_1(x, y) G(u^{(l)}_{m-1}(x, y)) \, dy + \sum_{j=0}^{k-1} \int_a^x k_2(x, y) D^j(u_{m-1}(x, y)) \, dy. \quad (2.35)\]

Substituting Eq.(2.35) into Eq.(2.33)

\[L[u_m(x, y) - \chi_m u_{m-1}(x, y)] = hH(x, y)[u_{m-1}(x, y) - \int_a^x k_1(x, y) G(u^{(l)}_{m-1}(x, y)) \, dy + \sum_{j=0}^{k-1} \int_a^x k_2(x, y) D^j(u_{m-1}(x, y)) \, dy]. \quad (2.36)\]
We take an initial guess \( u_0(x, y) = F_1(x, y) \), an auxiliary linear operator \( Lu = u \), a nonzero auxiliary parameter \( h = -1 \), and auxiliary function \( H(x, y) = 1 \). This is substituted into Eq. (2.36) to give the recurrence relation

\[
    u_0(x, y) = F_1(x, y),
    \]

\[
    u_n(x, y) = \int_a^x k_1(x, y) \ G(u_{n-1}^{(l)}(x, y)) \ dy - \sum_{j=0}^{k-1} \int_a^x k_2(x, y) \ D^j(u_{n-1}(x, y)) \ dy \quad n \geq 1.
\]

(2.37)

### 2.4 The modified homotopy perturbation method

First review homotopy perturbation method, we consider the following nonlinear differential equation:

\[
    A(\nu) - f(r) = 0 \quad r \in \Omega,
\]

(2.38)

with the boundary conditions:

\[
    B(\nu, \frac{\partial \nu}{\partial n}) = 0 \quad r \in \Gamma,
\]

(2.39)

where \( A \) is a general differential operator, \( B \) is a boundary operator, \( f(r) \) is a known analytical function and \( \Gamma \) is the boundary of the domain \( \Omega \). Generally speaking, operator \( A \) can be divided into two parts which are \( L \) and \( N \) where \( L \) is linear, but \( N \) is nonlinear. Therefore equation Eq. (2.38) can be rewritten as follows:

\[
    L(\nu) + N(\nu) - f(r) = 0.
\]

(2.40)

By the homotopy perturbation technique, we construct a homotopy \( \nu(r, p) : \Omega \times [0, 1] \to R \) which satisfies:

\[
    H(\nu, p) = (1 - p)[L(\nu) - L(u_0)] + p[A(\nu) - f(r)]
    \]

\[
    = L(\nu) - (1 - p)L(u_0) + p[N(\nu) - f(r)]
    \]

\[
    = 0, \quad p \in [0, 1], \quad r \in \Omega,
\]

(2.41)

where \( p \in [0, 1] \) is an embedding parameter, \( u_0 \) is an initial approximation which satisfies the boundary conditions. Therefore, obviously we have:

\[
    H(\nu, 0) = L(\nu) - L(u_0) = 0
    \]

\[
    H(\nu, 1) = A(\nu) - f(r) = 0
\]

Changing the process of \( p \) from zero to unity is just that of \( \nu(r, p) \) from \( u_0(r) \) to \( \nu(r) \). In topology, this is called deformation and \( L(\nu) - L(u_0) \) and \( A(\nu) - f(r) \) are called homotopy. According to the HPM, we can first use the embedding parameter \( p \) as a small parameter and assuming that the solution of Eq. (2.41) can be written as a power series in \( p \):

\[
    \nu = \nu_0 + p\nu_1 + p^2\nu_2 + \cdots
\]

(2.42)

setting \( p = 1 \), results in the approximate solution of Eq. (2.38):

\[
    \nu = \lim_{p \to 1} \nu = \nu_0 + \nu_1 + \nu_2 + \cdots
\]

(2.43)

In this paper, we don’t explain the modified homotopy perturbation method (MHPM), because the complete detail of the method are found in [21, 34].
2.4.1 The application of MHPM

In this section, the extended the modified homotopy perturbation method [39, 46] is used to find approximate of two-dimensional nonlinear Volterra-Fredholm integro-differential equation Eq. (1.1), according to the MHPM, we have

\[ L(u) = u(x,y) - F_1(x,y) - \int_a^x k_1(x,y) G(u^{(l)}(x,y)) \ dy + \sum_{j=0}^{k-1} \int_a^x k_2(x,y) D^j(u(x,y)) \ dy, \]

where \( D^j(u(x,y)) = g_1(x)h_1(y) \) and \( G(u(x,y)) = g_2(x)h_2(y) \). We can define homotopy \( H(u,p,m) \) by

\[ H(u,o,m) = f(u), \quad H(u,1,m) = L(u). \]

where \( m \) is an unknown real number and

\[ f(u(x,y)) = u(x,y) - F_1(x,y). \]

Typically we may choose a convex homotopy by

\[ H(u,p,m) = (1-p)f(u) + pL(u) + p(1-p)[m(g_1(x) + g_2(x))] = 0, \quad 0 \leq p \leq 1. \tag{2.44} \]

where \( m \) is called the accelerating parameters, and for \( m = 0 \) we define \( H(u,p,0) = H(u,p) \), which is the standard HPM. The convex homotopy Eq.(2.44) continuously trace an implicity defined curve from a starting point \( H(u(x,y) - f(u(x,y))), 0, m \) to a solution function \( H(u(x,y), 1, m) \). The embedding parameter \( p \) monotonically increase from 0 to 1 as trivial problem \( f(u) = 0 \) is continuously deformed to original problem \( L(u) = 0 \). [19, 32]

The MHPM uses the homotopy parameter \( p \) as an expanding parameter to obtain

\[ v = \sum_{n=0}^{\infty} p^n u_n, \tag{2.45} \]

when \( p \to 1 \) Eq. (2.45) corresponds to the original one, Eq. (2.44) becomes the approximate solution of Eq.(1.1) Eq.(1), i.e.,

\[ u = \lim_{p \to 1} v = \sum_{n=0}^{\infty} u_n, \tag{2.46} \]

where,

\[
\begin{align*}
    u_0(x,y) & = F_1(x,y), \\
    u_1(x,y) & = \int_a^x k_1(x,y) G(u_0^{(l)}(x,y)) \ dy \\
              & - \sum_{j=0}^{k-1} \int_a^x k_2(x,y) D^j(u_0(x,y)) \ dy - m(g_1(x) + g_2(x)), \\
    u_2(x,y) & = \int_a^x k_1(x,y) G(u_1^{(l)}(x,y)) \ dy \\
              & - \sum_{j=0}^{k-1} \int_a^x k_2(x,y) D^j(u_1(x,y)) \ dy + m(g_1(x) + g_2(x)), \\
    & \vdots \\
    u_m(x,y) & = \sum_{k=0}^{m-1} \int_a^x k_1(x,y) G(u_{m-1}^{(l)}(x,y)) \ dy \\
              & - \sum_{j=0}^{k-1} \int_a^x k_2(x,y) D^j(u_{m-1}(x,y)) \ dy, \quad m \geq 3.
\end{align*}
\]
Remark 2.1. Similarly, we can use these methods for two-dimensional nonlinear Volterra-Fredholm integro-differential as follows:

\[
\sum_{j=0}^{k} p_j(x,y)u^{(j)}_{y}(x,y) = f(x,y)+ \int_{a}^{y} \int_{\phi} k(x,y) G(u^{(l)}(x,y)) \, dx \, dy, \quad (x,y) \in \mathcal{J} = [a, y] \times \phi.
\]

3 Existence solution and convergence of iterative methods

In this section the existence and uniqueness of the obtained solution and convergence of the methods are proved. Consider the Eq. (1.7), we assume \( F_{1}(x,y) \) is bounded for all \( x, y \) in \( \mathcal{J} \) and

\[
|k_1(x,y)| \leq N_1, \\
|k_2(x,y)| \leq N_{1j}, \quad j = 0, 1, \ldots, k - 1, \quad \forall x, y \in \mathcal{J}.
\]

Also, we suppose the nonlinear terms \( G(u^{(l)}(x,y)) \) and \( D^j(u(x,y)) \) are Lipschitz continuous with

\[
|G(u^{(l)}(x,y)) - G(u^{(l)}(x,y))| \leq d \, u(x,y) - u^*(x,y), \\
|D^j(u(x,y)) - D^j(u^*(x,y))| \leq Z_j \, u(x,y) - u^*(x,y), \quad j = 0, 1, \ldots, k - 1.
\]

If we set,

\[
\gamma = (b - a) (d \, N_1 + k \, Z \, N), \\
Z = \max \{ Z_j \}, \quad N = \max \{ N_{1j} \}, \quad j = 0, 1, \ldots, k - 1.
\]

Then the following theorems can be proved by using the above assumptions.

Theorem 3.1. Two-dimensional nonlinear Volterra-Fredholm integro-differential equation, has a unique solution whenever \( 0 < \gamma < 1 \).

Proof. Let \( u \) and \( u^* \) be two different solutions of Eq. (1.7) then

\[
|u(x,y) - u^*(x,y)| = |\int_{a}^{x} k_1(x,y) \left[ G(u^{(l)}(x,y)) - G(u^{(l)}(x,y)) \right] \, dyt \\
- \sum_{j=0}^{k-1} \int_{a}^{x} k_2(x,y) \left[ D^j(u(x,y)) - D^j(u^*(x,y)) \right] \, dy | \\
\leq \int_{a}^{x} |k_1(x,y)| | G(u^{(l)}(x,y)) - G(u^{(l)}(x,y)) | \, dy \\
+ \sum_{j=0}^{k-1} \int_{a}^{x} |k_2(x,y)|| D^j(u(x,y)) - D^j(u^*(x,y)) | \, dy \leq (b - a) (d \, N_1 + k \, Z \, N) | u(x,y) - u^*(x,y) | \\
= \gamma | u(x,y) - u^*(x,y) | .
\]

from which we get \((1-\gamma)|u - u^*| \leq 0\). Since \( 0 < \gamma < 1 \), so \( |u - u^*| = 0 \). therefore, \( u = u^* \) and this completes the proof.

Theorem 3.2. The series solution \( u(x,y) = \sum_{i=0}^{\infty} u_i(x,y) \) of Eq. (1.1) using MADM convergence when \( 0 < \gamma < 1 \) and \( \| u_1(x,y) \| < \infty \).
Proof. Denote as \((C[J], \| \cdot \|)\) the Banach space of all continuous functions on \(J\) with the norm \(\| f(x, y) \| = \max |f(x, y)|\) for all \(x, y\) in \(J\). Define the sequence of partial sums \(s_n\), let \(s_n\) and \(s_m\) be arbitrary partial sums with \(n \geq m\). We are going to prove that \(s_n = \sum_{i=0}^{n} u_i(x, t)\) is a Cauchy sequence in this Banach space:

\[
\| s_n - s_m \| = \max_{\forall x, y \in J'} \| s_n - s_m \|
\]

\[
\| s_n - s_m \| = \max_{\forall x, y \in J'} \left| \sum_{i=m+1}^{n} u_i(x, y) \right|
\]

\[
= \max_{\forall x, y \in J'} \left[ \sum_{i=m+1}^{n} \left[ \int_{a}^{x} k_1(x, y) A_i \, dy \right. \right.
\]

\[
- \sum_{j=0}^{k-1} \int_{a}^{x} k_2(x, y) L_j \, dy \left. \right] \right|
\]

\[
= \max_{\forall x, y \in J'} \left| \int_{a}^{x} k_1(x, y) \left( \sum_{i=m}^{n-1} A_i \right) \, dy - \sum_{j=0}^{k-1} \int_{a}^{x} k_2(x, y) \left( \sum_{i=m}^{n-1} L_j \right) \, dy \right|.
\]

From [12], we have

\[
\sum_{i=m}^{n-1} A_i = G(s_{n-1}) - G(s_m-1),
\]

\[
\sum_{i=m}^{n-1} L_i = D^j(s_{n-1}) - D^j(s_m-1).
\]

So,

\[
\| s_n - s_m \| = \max_{\forall x, y \in J'} \left| \int_{a}^{x} k_1(x, y) \| G(s_{n-1}) - G(s_m-1) \| \, dy \right.
\]

\[
- \sum_{j=0}^{k-1} \int_{a}^{x} k_2(x, y) \| D^j(s_{n-1}) - D^j(s_m-1) \| \, dy \left. \right| \right|
\]

\[
\leq \max_{\forall x, y \in J'} \left( \int_{a}^{x} \| k_1(x, y) \| \cdot \| G(s_{n-1}) - G(s_m-1) \| \, dy \right)
\]

\[
+ \sum_{j=0}^{k-1} \left( \int_{a}^{x} \| k_2(x, y) \| \cdot \| D^j(s_{n-1}) - D^j(s_m-1) \| \, dy \right)
\]

\[
\leq \gamma \| s_{n-1} - s_{m-1} \| .
\]

Let \(n = m + 1\), then

\[
\| s_n - s_m \| \leq \gamma \| s_m - s_{m-1} \| \leq \gamma^2 \| s_{m-1} - s_{m-2} \| \leq \cdots \leq \gamma^m \| s_1 - s_0 \| .
\]

so,

\[
\| s_n - s_m \| \leq \| s_{m+1} - s_m \| + \| s_{m+2} - s_{m+1} \| + \cdots + \| s_n - s_{n-1} \|
\]

\[
\leq \left[ \gamma^m + \gamma^{m+1} + \cdots + \gamma^{n-m-1} \right] \| s_1 - s_0 \|
\]

\[
\leq \gamma^m \left[ 1 + \gamma + \gamma^2 + \cdots + \gamma^{n-m-1} \right] \| s_1 - s_0 \|
\]

\[
\leq \left[ \frac{1 - \gamma^{n-m}}{1 - \gamma} \right] \| u_1(x, y) \| .
\]

Since \(0 < \gamma < 1\), we have \((1 - \gamma^{n-m}) < 1\), then

\[
\| s_n - s_m \| \leq \frac{\gamma^m}{1 - \gamma} \| u_1(x, y) \| .
\]

But \(| u_1(x, y) | < \infty \) (since \(F_1(x, y)\) is bounded), so, as \(m \to \infty\), then \(\| s_n - s_m \| \to 0\). We conclude that \(s_n\) is a Cauchy sequence in \(C[J']\), therefore the series is convergence and the proof is complete. \(\Box\)
Theorem 3.3. When using VIM for solving two-dimensional nonlinear Volterra-Fredholm integro-differential equation that $0 < \gamma < 1$ and $p_k(x, y) = 1$ then $u(x, y) = \lim_{n \to \infty} u_n(x, y)$ is converges.

Proof. 

\[
\begin{align*}
u_{n+1}(x, y) &= u_n(x, y) - \int_0^y [u_n(x, \tau) - F_1(x, \tau)] d\tau \\
&- \int_a^x k_1(x, \tau) G(u^{(l)}(u_n(x, \tau), x)) d\tau - \sum_{j=0}^{k-1} \int_a^x k_2(x, \tau) (u_n)^{(j)}(x, \tau) d\tau \\
u(x, y) &= u(x, y) - \int_0^y [u(x, \tau) - F_1(x, \tau)] d\tau \\
&- \int_a^x k_1(x, \tau) G(u^{(l)}(u(x, \tau), x)) d\tau - \sum_{j=0}^{k-1} \int_a^x k_2(x, \tau) (u)^{(j)}(x, \tau) d\tau
\end{align*}
\]

(3.48)

(3.49)

By subtracting relation Eq. (3.48) from Eq. (3.49),

\[
u_{n+1}(x, y) - u(x, y) = u_n(x, y) - u(x, y) - \int_0^y [u_n(x, \tau) - u(x, \tau)] d\tau \\
&- \int_a^x k_1(x, \tau) [G(u^{(l)}(u_n(x, \tau), x)) - G(u^{(l)}(u(x, \tau), x))] d\tau \\
&- \sum_{j=0}^{k-1} \int_a^x k_2(x, \tau) [D^j(u_n(x, \tau) - D^j(u(x, \tau)))] d\tau \\
&- \{e_n(x, y) - e_n(x_0, y_0)\}
\]

If we set, $e_{n+1}(x, y) = u_{n+1}(x, y) - u(x, y)$, $e_n(x, y) = u_n(x, y) - u(x, y)$ then

\[
e_{n+1}(x, y) = e_n(x, y) - \int_0^y [u_n(x, \tau) - u(x, \tau)] d\tau \\
&- \int_a^x k_1(x, \tau) [G(u^{(l)}(u_n(x, \tau), x)) - G(u^{(l)}(u(x, \tau), x))] d\tau \\
&- \sum_{j=0}^{k-1} \int_a^x k_2(x, \tau) [D^j(u_n(x, \tau) - D^j(u(x, \tau)))] d\tau \\
&- \{e_n(x, y) - e_n(x_0, y_0)\}
\]

\[
\leq e_n(x, y)(1 - (b - a) (d N_1 + k Z N))
\]

\[
= (1 - \gamma)e_n(x, y)
\]

therefore,

\[
\| e_{n+1} \| = \max_{\mathcal{V}_x,y,j} | e_{n+1} | \\
\leq (1 - \gamma)\max_{\mathcal{V}_x,y,j} | e_n | \\
= \| e_n \|.
\]

(3.51)

Since $0 < \gamma < 1$, then $\| e_n \| \to 0$. So, the series converges and the proof is complete.

Theorem 3.4. If the series solution Eq. (2.38) of problem Eq. (1.1) is convergent then it converges to the exact solution of the problem Eq. (1.1) by using HAM.

Proof. We assume:

\[
\hat{G}(u^{(l)}(x, y)) = \sum_{m=0}^\infty G(u_m(x, y)),
\]

\[
\hat{D}^j(u(x, y)) = \sum_{m=0}^\infty D^j(u_m(x, y)),
\]

\[
u(x, y) = \sum_{m=0}^\infty u_m(x, y),
\]
where,
\[ \lim_{m \to \infty} u_m(x, y) = 0. \]

We can write,
\[
\sum_{m=1}^{n} [u_m(x, y) - \chi_m u_{m-1}(x, y)] = u_1 + (u_2 - u_1) + \cdots + (u_n - u_{n-1}) = u_n(x, y). \tag{3.52}
\]

Hence, from Eq. (3.52)
\[ \lim_{n \to \infty} u_n(x, y) = 0. \tag{3.53} \]

So, using Eq. (3.53) and the definition of the linear operator \(L\), we have
\[
\sum_{m=1}^{\infty} L[u_m(x, y) - \chi_m u_{m-1}(x, y)] = L[\sum_{m=1}^{\infty} [u_m(x, y) - \chi_m u_{m-1}(x, y)]] = 0.
\]

Therefore from Eq. (2.34), we can obtain that,
\[
\sum_{m=1}^{\infty} L[u_m(x, y) - \chi_m u_{m-1}(x, y)] = hH(x, y) \sum_{m=1}^{\infty} \Re_{m-1}(u_{m-1}(x, y)) = 0. \tag{3.54}
\]

Since \(h \neq 0\) and \(H(x, y) \neq 0\), we have
\[
\sum_{m=1}^{\infty} \Re_{m-1}(u_{m-1}(x, y)) = 0. \tag{3.55}
\]

By substituting \(\Re_{m-1}(u_{m-1}(x, y))\) into the relation Eq. (3.55) and simplifying it, we have
\[
\sum_{m=1}^{\infty} \Re_{m-1}(u_{m-1}(x, y)) = \sum_{m=1}^{\infty} [u_{m-1}(x, y) - \int_{a}^{x} k_1(x, y) G(u_{m-1}^{(l)}(x, y)) dy]
- \sum_{j=1}^{k-1} \int_{a}^{x} k_2(x, y) D^j(u_{m-1}(x, y)) dy - (1 - \chi_m)F_1(x, y)
= u(x, y) - F_1(x, y) - \int_{a}^{x} k_1(x, y) \left[ \sum_{m=1}^{\infty} G(u_{m-1}^{(l)}(x, y)) \right] dy
- \sum_{j=1}^{k-1} \int_{a}^{x} k_2(x, y) \left[ \sum_{m=1}^{\infty} D^j(u_{m-1}(x, y)) \right] dy.
\tag{3.56}
\]

From Eq. (3.55) and Eq. (3.56), we have
\[ u(x, y) = F_1(x, t) + \int_{a}^{x} k_1(x, y) G(u_{m-1}^{(l)}(x, y)) dy - \sum_{j=0}^{k-1} \int_{a}^{x} k_2(x, y) D^j(u(x, y)) dy, \]
therefore, \(u(x, y)\) must be the exact solution of Eq. (1.1). \(\square\)

**Theorem 3.5.** The series solution \(u(x, y) = \sum_{i=0}^{\infty} u_i(x, y)\) of Eq. (1.7) using MHPM converges [43].

### 4 Numerical examples

In this section, we compute numerical examples which are solved by the MADM, VIM, HAM and MHPM. The program has been provided with Mathematica 6.
Example 4.1. Consider the two-dimensional nonlinear Volterra-Fredholm integro-differential equation

\[ u''(x,t) + \sin(\pi t) u(x,t) = \pi t \sin(\pi t) \frac{1}{3} t^3 + \int_0^t \int_0^1 \pi t u'(x,t) \, dx \, dt, \]

with the initial conditions

\[ u_x(0,t) = t, \quad u(0,t) = 0. \]

The exact solution is \( u(x,t) = \pi t \). Also, \( \varepsilon = 10^{-2} \).

Table 1

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>Errors (MADM, (n=10))</th>
<th>Errors (VIM, (n=7))</th>
<th>Errors (MHPM, (n=5))</th>
<th>Errors (HAM, (n=3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.1, 0.05)</td>
<td>0.074634</td>
<td>0.053367</td>
<td>0.042579</td>
<td>0.032336</td>
</tr>
<tr>
<td>(0.2, 0.14)</td>
<td>0.074727</td>
<td>0.054175</td>
<td>0.043259</td>
<td>0.032673</td>
</tr>
<tr>
<td>(0.4, 0.23)</td>
<td>0.075597</td>
<td>0.055845</td>
<td>0.045384</td>
<td>0.034252</td>
</tr>
<tr>
<td>(0.6, 0.27)</td>
<td>0.077224</td>
<td>0.057427</td>
<td>0.046647</td>
<td>0.035566</td>
</tr>
<tr>
<td>(0.85, 0.35)</td>
<td>0.078538</td>
<td>0.058762</td>
<td>0.048345</td>
<td>0.037331</td>
</tr>
</tbody>
</table>

Example 4.2. Consider the following equation given by

\[ u''(x,t) + x u'''(x,t) = 4\pi e^t + \int_0^t \int_0^1 e^{\pi t^2} u''''(x,t) \, dx \, dt, \]

subject to the initial conditions

\[ u_x(0,t) = u(0,t) = 0. \]

The exact solution is \( u(x,t) = x^2 e^t \). \( \varepsilon = 10^{-3} \).

Table 2

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>Errors (MADM, (n=17))</th>
<th>Errors (VIM, (n=12))</th>
<th>Errors (MHPM, (n=8))</th>
<th>Errors (HAM, (n=6))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.15, 0.10)</td>
<td>0.0082732</td>
<td>0.0062078</td>
<td>0.0051238</td>
<td>0.0042345</td>
</tr>
<tr>
<td>(0.28, 0.27)</td>
<td>0.0083451</td>
<td>0.0062653</td>
<td>0.0051782</td>
<td>0.0042755</td>
</tr>
<tr>
<td>(0.56, 0.33)</td>
<td>0.0085128</td>
<td>0.0064207</td>
<td>0.0053341</td>
<td>0.0044883</td>
</tr>
<tr>
<td>(0.65, 0.42)</td>
<td>0.0085892</td>
<td>0.0064894</td>
<td>0.0053907</td>
<td>0.0045275</td>
</tr>
<tr>
<td>(0.86, 0.58)</td>
<td>0.0087015</td>
<td>0.0066239</td>
<td>0.0055346</td>
<td>0.0047109</td>
</tr>
</tbody>
</table>

Tables 1 and 2 show that, the error of the HAM is less than the error of the MADM, VIM and MHPM.

5 Conclusion

The HAM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which are convergent are rapidly to the exact solutions. In this work, the HAM has been successfully employed to obtain the approximate solution to analytical solution of the two-dimensional nonlinear Volterra-Fredholm integro-differential equation. For this purpose in examples, we showed that the HAM is more rapid convergence than the MADM, VIM and MHPM.
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