On the solution of the Generalized Bretherton Equation by the Homogeneous Balance Method

Cem Oğuz1, Ahmet Yıldırım2,3,*, Shahlar Meherrem1
(1) Department of Mathematics, Yasar University, Bornova -Izmir, Turkey
(2) Department of Mathematics, Science Faculty, Ege University, 35100 Bornova-Izmir, Turkey
(3) University of South Florida, Department of Mathematics and Statistics, Tampa, FL 33620-5700, USA

Abstract
In this paper, we try to find exact travelling wave solutions of the generalized Bretherton equation for integer, greater than one, values of the exponent $m$ of the nonlinear term. We will use the homogeneous balance method (HBM) to solve the Riccati equation and the reduced nonlinear ordinary differential equation to obtain solutions of the generalized Bretherton equation.

Keywords: Bretherton equation; traveling wave solutions; homogeneous balance method.

1 Introduction
In recent years, Wang et al. [12, 13, 14] presented a useful homogeneous balance method (HBM) for finding exact solutions of given nonlinear partial differential equations. Some authors used improved HBM and made straighter forward and simple [7, 10, 18, 17]. However, they are restricted because the difficulty of finding solutions of Ansatz equations, such as the Riccati equation, etc. In this paper, we use the homogeneous balance method (HBM) to solve the Riccati equation and the reduced nonlinear ordinary differential equation, respectively. We choose the generalized Bretherton equation to illustrate efficiency of the homogeneous balance method [1, 16, 5].

The aim of this paper to search periodic and solitary travelling wave solutions of the generalized Bretherton equation [11]

$$u_{tt} + u_{xx} + u_{xxxx} + u - u^m,$$  
(1.1)

for integer values of $m \geq 2$.

For $m = 2$, we get the partial differential equation in time and one spatial dimension where $u$ is a real function, was introduced by Bretherton [3] as a model of a dispersive wave system to investigate the resonant nonlinear interaction between three linear modes.

*Corresponding author. Email address: ahmetyildirim80@gmail.com; Tel:+90(535)9233879.
For $m = 3$, we get the modified Bretherton equation, this equation was considered by some authors as an example of application of methods to obtain travelling wave solutions of nonlinear wave equations [8, 2, 4, 15, 6, 9]. Kudryashov [8] used a truncated Painlevé expansion and an auxiliary equation method to obtain a periodic solution in terms of elliptic functions. Berloff and Howard [2] also used a truncated Painlevé expansion as a starting point to search for a periodic solution as an infinite sum of translated sech squared functions.

Recently, Romeiras [11] used the truncated Painlevé expansion method and an algebraic method to find periodic solutions of the generalized Bretherton equation. The present paper uses the homogeneous balance method (HBM) to search for exact travelling wave solutions of the generalized Bretherton equation for integer $m$, greater than one, values of the exponent $m$ of the nonlinear term.

Firstly, we assume the solutions of Eq. (1.1) in this form,

$$u(x, t) = u(\xi), \quad \xi = kx - ct,$$

then we obtain the ordinary differential equation

$$mu_{\xi\xi\xi\xi} + nu_{\xi\xi} + u - u^m = 0,$$

where $m = k^4$, $n = c^2 + k^2$.

## 2 Solutions by the homogeneous balance method (HBM)

**Example 1:** Consider Eq. (1.3) for $m = 2$,

$$mu_{\xi\xi\xi\xi} + nu_{\xi\xi} + u - u^2 = 0.$$

We seek the solution of Eq. (2.4) in the form

$$u = \sum_{i=0}^{N} a_i \varphi^i,$$

where $a_i$ are constants to be determined later and $\varphi$ satisfy the Riccati equation

$$\varphi' = a\varphi^2 + c,$$

where $a$ and $c$ are constants and $\varphi$ satisfies the Riccati equation. If balancing $u_{\xi\xi\xi\xi}$ with $u^2$ in Eq. (2.4), it is easy to show that $N = 4$. Therefore use the ansatz

$$u = a_0 + a_1 \varphi + a_2 \varphi^2 + a_3 \varphi^3 + a_4 \varphi^4.$$

Substituting Eq. (2.7) into Eq. (2.4) along with (2.6) and collecting all terms with the same power in $\varphi^i$ ($i = 0, 1, \ldots, 8$) yields a set of algebraic system for $a_0, a_1, a_2, a_3, a_4, n$ and $m$, namely
16ma^3a_2 + 24ma^4a_4 + 2na_2c^2 + a_0 - a_0^2 = 0, \\
16ma^2c^2a_1 + 120mac^3a_3 + 6na_3c^2 + 2na_1ac - 2aq_0a_1 + a_1 = 0, \\
136ma^2c^2a_2 + 12nc^2a_4 + 480ma_4c^3a + 8na_2ac - 2aq_0a_2 + a_2 - a_1^2 = 0, \\
576ma^2c^3a_3 + 40ma_1a_3 + 2na_1a_2^2 + 18na_3ac - 2aq_0a_3 - 2a_1a_2 - a_3 = 0, \\
240ma^3ca_2 + 1696ma_4c^2a^2 + 6na_2a^2 + 32na_4ac - 2aq_0a_4 - 2a_1a_3 + a_4 - a_2^2 = 0, \quad (2.8) \\
816ma^3ca_3 + 24ma_1a^4 + 12na_3a^2 - 2aq_0a_4 - 2a_1a_3 = 0, \\
2080ma^3ca_4 + 120ma_2a^4 + 20na_4a^2 - 2aq_0a_4 - a_3^2 = 0, \\
360ma^4a_3 - 2a_3a_4 = 0, \\
840ma^4a_4 - a_4^2 = 0.

For which, with the aid of Maple, we find

\[
a_0 = \frac{35}{24}, \quad a_1 = 0, \quad a_2 = \frac{35a}{12c}, \quad a_3 = 0, \quad a_4 = \frac{35a^2}{24c^2},
\]

and

\[
a_0 = -\frac{11}{24}, \quad a_1 = 0, \quad a_2 = -\frac{35a}{12c}, \quad a_3 = 0, \quad a_4 = -\frac{35a^2}{24c^2}.
\]

It is to be noted that the Riccati equation (2.6) can be solved using the homogeneous balance method as follows:

**Type 1:** Let \( \varphi = \sum_{i=0}^{N} b_i \tanh^i \xi \). Balancing \( \varphi \) with \( \varphi^2 \) leads to

\[
\varphi = b_0 + b_1 \tanh \xi.
\]

Substituting Eq. (2.11) into (2.6), we have the following solution of Eq. (2.6)

\[
\varphi = -\frac{1}{a} \tanh \xi = c \tanh \xi,
\]

where \( ac = -1 \).

From (2.7), (2.9), (2.10) and (2.12), we have the following traveling wave solutions of Eq. (2.4):

\[
\begin{align*}
    u_1(x,t) &= \frac{35}{24} + \frac{35ac}{12} \tanh^2(kx - ct) + \frac{35a^2c^2}{24} \tanh^4(kx - ct), \\
    u_2(x,t) &= -\frac{11}{24} - \frac{35ac}{12} \tanh^2(kx - ct) - \frac{35a^2c^2}{24} \tanh^4(kx - ct).
\end{align*}
\]

Similarly, let \( \varphi = \sum_{i=0}^{N} b_i \coth^i \xi \), then we obtain the following new traveling wave soliton solutions of Eq. (2.4)

\[
\begin{align*}
    u_3(x,t) &= \frac{35}{24} + \frac{35ac}{12} \coth^2(kx - ct) + \frac{35a^2c^2}{24} \coth^4(kx - ct), \\
    u_4(x,t) &= -\frac{11}{24} - \frac{35ac}{12} \coth^2(kx - ct) - \frac{35a^2c^2}{24} \coth^4(kx - ct).
\end{align*}
\]

**Type 2:** From [17], when \( a = 1, \ b = 0 \), the Riccati equation (2.6) has the following solutions

\[
\varphi = \begin{cases}
    -\sqrt{-c} \tanh(\sqrt{-c} \xi), & c < 0 \\
    -1/\xi, & c = 0 \\
    \sqrt{c} \tanh(\sqrt{c} \xi), & c > 0
\end{cases}
\]

(2.15)
From (2.7), (2.9), (2.10) and (2.15), we have the following traveling wave solutions of Eq. (2.4) as follows:

When $c < 0$, we have

$$u_1(x, t) = \frac{35}{24} - \frac{35a}{12} \tanh^2(\sqrt{-c}(kx - ct)) + \frac{35a^2}{24} \tanh^4(\sqrt{-c}(kx - ct)),
$$

$$u_2(x, t) = -\frac{11}{24} + \frac{35a}{12} \tanh^2(\sqrt{-c}(kx - ct)) - \frac{35a^2}{24} \tanh^4(\sqrt{-c}(kx - ct)).$$

(2.16)

When $c = 0$, we have

$$u_3(x, t) = \frac{35}{24} + \frac{35a}{12} \tanh(\sqrt{c}(kx - ct)) + \frac{35a^2}{24} \tanh^4(\sqrt{c}(kx - ct)),
$$

$$u_4(x, t) = -\frac{11}{24} - \frac{35a}{12} \tanh(\sqrt{c}(kx - ct)) - \frac{35a^2}{24} \tanh^4(\sqrt{c}(kx - ct)).$$

(2.17)

When $c > 0$, we have

$$u_5(x, t) = \frac{35}{24} + \frac{35a}{12} \tan^2(\sqrt{c}(kx - ct)) + \frac{35a^2}{24} \tan^4(\sqrt{c}(kx - ct)),
$$

$$u_6(x, t) = -\frac{11}{24} - \frac{35a}{12} \tan^2(\sqrt{c}(kx - ct)) - \frac{35a^2}{24} \tan^4(\sqrt{c}(kx - ct)).$$

(2.18)

**Type 3**: We suppose that the Riccati equation (2.6) has the following solutions of the form

$$\varphi = A_0 + \sum_{i=1}^{N} (A_i f^i + B_i f^{i-1}g),$$

(18)

with

$$f = \frac{1}{\cosh \xi + r}, \quad g = \frac{\sinh \xi}{\sinh \xi + r},$$

(2.19)

which satisfy

$$f'(\xi) = -f(\xi)g(\xi), \quad g'(\xi) = 1 - g^2(\xi) - rf(\xi).$$

Balancing $\varphi$ with $\varphi^2$ leads to

$$\varphi = A_0 + A_1 f + B_1 g.$$  

(2.20)

Substituting Eq. (2.20) into (2.6), collecting the coefficient of the same power $f^i(\xi), g^j(\xi)$ ($i = 0, 1, 2; j = 0, 1$) and setting each of the obtained coefficients to zero yield the following set of algebra equations

$$aA_1^2 + a(r^2 - 1)B_1^2 + (r^2 - 1)B_1 + = 0,$$

$$2aB_1 A_1 + A_1 = 0,$$

$$2aA_1 A_0 - 2arB_1^2 - rB_1 = 0,$$

$$aA_0^2 + aB_1^2 + c = 0,$$

$$2aB_1 A_0 = 0,$$

which have solutions

$$A_0 = 0, \quad A_1 = \pm \sqrt{\frac{(r^2 - 1)}{4a^2}}, \quad B_1 = \frac{-1}{2a}, \quad c = \frac{-1}{4a}.$$  

(2.21)
From (2.20)–(2.21), we have
\[ \varphi = -\frac{1}{2a} \left( \frac{\sinh \xi \mp \sqrt{(r^2 - 1)}}{\cosh \xi + r} \right). \tag{2.22} \]

From Eqs. (2.7), (2.9), (2.10) and (2.22), we obtain
\begin{align*}
 u_1(x,t) &= \frac{35}{24} + \frac{35}{48a} \left( \frac{\sinh \xi \mp \sqrt{(r^2 - 1)}}{\cosh \xi + r} \right)^2 + \frac{35}{384a^2c^2} \left( \frac{\sinh \xi \mp \sqrt{(r^2 - 1)}}{\cosh \xi + r} \right)^4, \\
 u_2(x,t) &= -\frac{11}{24} - \frac{35}{48a} \left( \frac{\sinh \xi \mp \sqrt{(r^2 - 1)}}{\cosh \xi + r} \right)^2 - \frac{35}{384a^2c^2} \left( \frac{\sinh \xi \mp \sqrt{(r^2 - 1)}}{\cosh \xi + r} \right)^4, \tag{2.23}
\end{align*}

where \( \xi = kx - ct \).

**Type 4:** We take \( \varphi \) in Riccati equation being of the form
\[ \varphi = e^{p_1 \xi}p(z) + p_4(\xi), \tag{2.24} \]
where \( z = e^{p_2 \xi} + p_3 \), and \( p_1, p_2 \) and \( p_3 \) are constants to be determined.

Substituting (2.24) into (2.6), we have
\[ \varphi' - a\varphi^2 - c = p_2e^{(p_1 + p_2)\xi}p' - ae^{2p_1\xi}p^2 + (p_1 - 2ap_4)e^{p_1\xi}p + p_4' - ap_4^2 - c = 0. \tag{2.25} \]

Setting \( p_1 + p_2 = 2p_1 \), we get \( p_1 = p_2 \). If we assume that \( p_4 = \frac{p_1}{2a} \) and \( c = -\frac{p_1^2}{4a} \), then (2.25) becomes
\[ p_2p' - ap^2 = 0. \tag{2.26} \]

By solving Eq. (2.26), we have
\[ p = \frac{p_1}{az} = -\frac{p_1}{a(e^{p_1\xi} + p_3)}. \tag{2.27} \]

Substituting (2.27) and \( p_4 = \frac{p_1}{2a} \) into (2.24), we have
\[ \varphi = \frac{p_1}{2a} - \frac{p_1e^{p_1\xi}}{a(e^{p_1\xi} + p_3)}. \tag{2.28} \]

If \( p_3 = 1 \) in (2.28), we have
\[ \varphi = -\frac{p_1}{2a} \tanh \left( \frac{1}{2}p_1\xi \right). \tag{2.29} \]

If \( p_3 = -1 \) in (2.28), we have
\[ \varphi = -\frac{p_1}{2a} \coth \left( \frac{1}{2}p_1\xi \right). \tag{2.30} \]

From (2.7), (2.9), (2.10) and (2.28), we obtain the following new wave solutions of Eq. (2.4).
\[ u_1(x, t) = \frac{35}{24} + \frac{35a}{12c} \left( \frac{p_1}{2a} - \frac{p_1 e^{p_1 \xi}}{a(e^{p_1 \xi} + p_3)} \right)^2 + \frac{35a^2}{24c^2} \left( \frac{p_1}{2a} - \frac{p_1 e^{p_1 \xi}}{a(e^{p_1 \xi} + p_3)} \right)^4, \]
\[ u_2(x, t) = -\frac{11}{24} - \frac{35a}{12c} \left( \frac{p_1}{2a} - \frac{p_1 e^{p_1 \xi}}{a(e^{p_1 \xi} + p_3)} \right)^2 - \frac{35a^2}{24c^2} \left( \frac{p_1}{2a} - \frac{p_1 e^{p_1 \xi}}{a(e^{p_1 \xi} + p_3)} \right)^4. \] (2.31)

When \( p_3 = 1 \), we have from (2.29)
\[ u_3(x, t) = \frac{35}{24} + \frac{35a}{12c} \left( \frac{p_1}{2a} \tanh \left( \frac{1}{2} p_1 \xi \right) \right)^2 + \frac{35a^2}{24c^2} \left( \frac{p_1}{2a} \tanh \left( \frac{1}{2} p_1 \xi \right) \right)^4, \]
\[ u_4(x, t) = -\frac{11}{24} - \frac{35a}{12c} \left( \frac{p_1}{2a} \tanh \left( \frac{1}{2} p_1 \xi \right) \right)^2 - \frac{35a^2}{24c^2} \left( \frac{p_1}{2a} \tanh \left( \frac{1}{2} p_1 \xi \right) \right)^4. \] (2.32)

Clearly, (2.13) is the special case of (2.32) with \( p_1 = 2 \).

When \( p_3 = 1 \), we have from (2.30)
\[ u_5(x, t) = \frac{35}{24} + \frac{35a}{12c} \left( \frac{p_1}{2a} \coth \left( \frac{1}{2} p_1 \xi \right) \right)^2 + \frac{35a^2}{24c^2} \left( \frac{p_1}{2a} \coth \left( \frac{1}{2} p_1 \xi \right) \right)^4, \]
\[ u_6(x, t) = -\frac{11}{24} - \frac{35a}{12c} \left( \frac{p_1}{2a} \coth \left( \frac{1}{2} p_1 \xi \right) \right)^2 - \frac{35a^2}{24c^2} \left( \frac{p_1}{2a} \coth \left( \frac{1}{2} p_1 \xi \right) \right)^4. \] (2.33)

Clearly, (2.14) is the special case of (2.33) with \( p_1 = 2 \).

**Example 2:** Let’s consider Eq. (1.3) for \( m = 5 \)
\[ m u_{\xi\xi\xi\xi\xi} + nu_{\xi\xi} + u - u^5 = 0. \] (2.34)

We seek the solution of Eq. (2.34) in the form
\[ u = \sum_{i=0}^{N} q_i \varphi^i, \] (2.35)
where \( q_i \) are constants to be determined later and \( \varphi \) satisfy the Riccati equation
\[ \varphi' = a \varphi^2 + b \varphi + c, \] (2.36)
where \( a, b \) and \( c \) are constants and \( \varphi \) satisfies the Riccati equation. If balancing \( u_{\xi\xi\xi\xi\xi} \) with \( u^5 \) in Eq. (2.36), it is easy to show that \( N = 1 \). Therefore use the ansatz
\[ u = a_0 + a_1 \varphi. \] (2.37)

Substituting Eq. (2.37) into (2.34) along with (2.36) and collecting all terms with the same power in \( \varphi^i \) \((i = 0, 1, \ldots, 5)\) yields a set of algebraic system for \( a_0, a_1, n \) and \( m \), namely
\[
\begin{align*}
8mabc^2a_1 + mb^5c_1 + na_1bc + a_0 &- a_0^5 = 0, \\
22mab^2c_1 + 16ma^2c^2a_1 + ma_1b^3 + 2na_1ac + na_1b^2 - 5a_0^3a_1 + a_1 &= 0, \\
60ma^2bca_1 + 3na_1ab + 15ma_1b^3a - 10na_0a_1^2 &= 0, \\
50ma_1a^2b^2 + 40mca_1a^2 + 2na_1a^2 - 10a_0^2a_1^2 &= 0, \\
60ma_1^3b - 5a_0a_1^4 &= 0, \\
24ma_1^2a - a_1^5 &= 0.
\end{align*}
\]
For which, with the aid of Maple, we find
\[ a_0 = \frac{\pm b}{\sqrt{4ac - b^2}}, \quad a_1 = \frac{\pm 2}{\sqrt{4ac - b^2}}, \] (2.38)
and

\[ a_0 = \pm \frac{b}{\sqrt{b^2 - 4ac}}, \quad a_1 = \mp \frac{2}{\sqrt{b^2 - 4ac}}. \tag{2.39} \]

For the Riccati equation (2.36), we can solve it by using the homogeneous balance method as follows:

**Type 1:** For \( \varphi = \sum_{i=0}^{N} b_i \tanh^i \xi \), the Riccati equation has the following solution:

\[ \varphi = -\frac{1}{2a} (b + 2 \tanh \xi), \quad ac = \frac{b^2}{4} - 1. \tag{2.40} \]

From (2.37)-(2.39) and (2.40), we have the following new traveling wave solution of Eq. (2.34)

\[ u_1(x, t) = \pm \frac{b}{\sqrt{4ac - b^2}} \pm \frac{1}{a} \frac{b + 2 \tanh \xi}{\sqrt{4ac - b^2}}, \tag{2.41} \]

\[ u_2(x, t) = \pm \frac{b}{\sqrt{b^2 - 4ac}} \pm \frac{1}{a} \frac{b + 2 \tanh \xi}{\sqrt{b^2 - 4ac}}. \tag{2.42} \]

Similarly, let \( \varphi = \sum_{i=0}^{N} b_i \coth^i \xi \), then we obtain the following new traveling wave soliton solutions of Eq. (2.34)

\[ u_3(x, t) = \pm \frac{b}{\sqrt{4ac - b^2}} \pm \frac{1}{a} \frac{b + 2 \coth \xi}{\sqrt{4ac - b^2}}, \tag{2.43} \]

\[ u_4(x, t) = \pm \frac{b}{\sqrt{b^2 - 4ac}} \pm \frac{1}{a} \frac{b + 2 \coth \xi}{\sqrt{b^2 - 4ac}}. \tag{2.44} \]

where \( \xi = kx + wt \).

**Type 2:** From [3], when \( a=1 \), \( b = 0 \), the Riccati equation (2.36) has the following solutions

\[ \varphi = \begin{cases} 
-\sqrt{-c} \tanh(\sqrt{-c} \xi), & c < 0 \\
-1/\xi, & c = 0 \\
\sqrt{c} \tanh(\sqrt{c} \xi), & a > 0 
\end{cases} \tag{2.45} \]

When \( c < 0 \), we have

\[ u_1(x, t) = \pm \frac{\pm b}{\sqrt{4ac - b^2}} \pm \frac{2 \sqrt{-c} \tanh(\sqrt{-c} (kx - ct))}{\sqrt{4ac - b^2}}, \tag{2.46} \]

\[ u_2(x, t) = \pm \frac{b}{\sqrt{b^2 - 4ac}} \pm \frac{2 \sqrt{-c} \tanh(\sqrt{-c} (kx - ct))}{\sqrt{b^2 - 4ac}}. \]

When \( c = 0 \), we have

\[ u_3(x, t) = \pm \frac{b}{\sqrt{4ac - b^2}} \pm \frac{2}{(kx - ct) \sqrt{4ac - b^2}}. \tag{2.47} \]

\[ u_4(x, t) = \pm \frac{b}{\sqrt{b^2 - 4ac}} \pm \frac{2}{(kx - ct) \sqrt{b^2 - 4ac}}. \]

When \( c > 0 \), we have

\[ u_5(x, t) = \pm \frac{b}{\sqrt{4ac - b^2}} \pm \frac{2 \sqrt{c} \tanh(\sqrt{c} (kx - ct))}{\sqrt{4ac - b^2}}, \tag{2.48} \]
\[ u_6(x,t) = \pm \frac{b}{\sqrt{b^2 - 4ac}} \mp \frac{2\sqrt{c} \tanh(\sqrt{c}(kx - ct))}{\sqrt{b^2 - 4ac}}. \quad (2.49) \]

**Type 3:** For the form \( \varphi = A_0 + \sum_{i=1}^{N} (A_i f^i + B_i f^{i-1} g) \), the Riccati equation has the solution

\[ \varphi = -\frac{1}{2a} \left( b + \frac{\sinh \xi \mp \sqrt{(r^2 - 1)}}{\cosh \xi + r} \right). \quad (2.50) \]

From (2.37)-(2.39) and (2.50), we obtain the new wave solutions of Eq. (2.34)

\[ u_1(x,t) = \pm \frac{b}{\sqrt{b^2 - 4ac}} \mp \frac{1}{a \sqrt{b^2 - 4ac}} \left( b + \frac{\sinh \xi \mp \sqrt{(r^2 - 1)}}{\cosh \xi + r} \right), \]
\[ u_2(x,t) = \pm \frac{b}{\sqrt{b^2 - 4ac}} \mp \frac{1}{a \sqrt{b^2 - 4ac}} \left( b + \frac{\sinh \xi \mp \sqrt{(r^2 - 1)}}{\cosh \xi + r} \right), \quad (2.51) \]

where \( \xi = kx - ct \).

**Type 4:** Taking \( \varphi \) in the Riccati equation being of the form

\[ \varphi = e^{p_1 \xi} p(z) + p_4(\xi), \]

we have the solutions as follows:

\[ \varphi = \frac{p_1 - b}{2a} - \frac{p_1 e^{p_1 \xi}}{a (p_1 e^{p_1 \xi} + p_3)}. \quad (2.53) \]

If \( p_3 = 1 \) in (2.53), we have

\[ \varphi = -\frac{p_1}{2a} \tanh \left( \frac{1}{2} p_1 \xi \right) - \frac{b}{2a}. \quad (2.54) \]

If \( p_3 = -1 \) in (2.53), we have

\[ \varphi = -\frac{p_1}{2a} \coth \left( \frac{1}{2} p_1 \xi \right) - \frac{b}{2a}. \quad (2.55) \]

From (2.37)-(2.39) and (2.53), we obtain the following new wave solutions of Eq. (2.34)

\[ u_1(x,t) = \pm \frac{b}{\sqrt{b^2 - 4ac}} \mp \frac{2}{a \sqrt{b^2 - 4ac}} \left( \frac{p_1 \mp b}{2a} - \frac{p_1 e^{p_1 \xi}}{a (p_1 e^{p_1 \xi} + p_3)} \right), \]
\[ u_2(x,t) = \pm \frac{b}{\sqrt{b^2 - 4ac}} \mp \frac{2}{a \sqrt{b^2 - 4ac}} \left( \frac{p_1 \mp b}{2a} - \frac{p_1 e^{p_1 \xi}}{a (p_1 e^{p_1 \xi} + p_3)} \right), \quad (2.56) \]

where \( \xi = kx - ct \).

When \( p_3 = 1 \), we have from (2.54)

\[ u_3(x,t) = \pm \frac{b}{\sqrt{b^2 - 4ac}} \mp \frac{2}{a \sqrt{b^2 - 4ac}} \left( \frac{p_1}{2a} \tanh \left( \frac{1}{2} p_1 \xi \right) + \frac{b}{2a} \right), \]
\[ u_4(x,t) = \pm \frac{b}{\sqrt{b^2 - 4ac}} \mp \frac{2}{a \sqrt{b^2 - 4ac}} \left( \frac{p_1}{2a} \tanh \left( \frac{1}{2} p_1 \xi \right) + \frac{b}{2a} \right). \quad (2.57) \]

When \( p_3 = -1 \), we have from (2.55)
\[ u_5(x, t) = \frac{\pm b}{\sqrt{4ac-b^2}} \pm \frac{2}{\sqrt{4ac-b^2}} \left( \frac{p_1}{2a} \coth \left( \frac{1}{2} p_1 \xi \right) + \frac{b}{2a} \right), \]
\[ u_6(x, t) = \frac{\pm b}{\sqrt{b^2-4ac}} \pm \frac{2}{\sqrt{b^2-4ac}} \left( \frac{p_1}{2a} \coth \left( \frac{1}{2} p_1 \xi \right) + \frac{b}{2a} \right). \] (2.58)

Clearly, (2.41), (2.42) and (2.43), (2.44) are the special case of (2.57) and (2.58) with \( p_1 = 2 \).

3 Conclusion

In this study, we have searched for exact solutions of the generalized Bretherton equation for \( m = 2 \) and \( m = 5 \). Firstly, we reduced the generalized Bretherton equation to nonlinear ordinary differential equation. Then we obtained exact travelling wave solutions by using Riccati equation and balancing. I think it would be interesting to apply this method to obtain travelling wave solutions to the partial differential equations.

Acknowledgement

Second author thanks TÜBİTAK (The Scientific and Technological Research Council of Turkey) for their financial support and grant for research entitled 'Integrable Systems and Soliton Theory' at University of South Florida.

References


