Error analysis for the solution of fuzzy differential equations using orthogonal basis functions

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Abstract

In this paper, the black pulse functions as a set of piecewise orthogonal functions are used to introduce a fuzzy solution for fuzzy differential equations. An error bound for the mentioned solution is investigated in details. To do this, 1-cut for of the FDEs is considered and after finding the solution and other related equations, the fuzzy models and equations are introduce after allocating the spreads.

Keywords: Black pulse, Error bound, Fuzzy models, Orthogonal functions.

1 Introduction

The fuzzy differential equations FDEs has an essential role in real world application science with fuzzy logic point of view. So many researchers have worked on FDEs find analytical and numerical solutions \([1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 16, 18]\). Finding error bound for the solution of FDEs is important task in application. To do this purpose we are going to find the error bound. The structure of the paper is as follow: In section two some definitions, notations and definitions are brought. In section three the main subject is discussed in section four an example is solved for more illustration of the theory. In section five the conclusion is brought.
2 Preliminaries

The basic definitions of a fuzzy number are given in [5, 14, 15] as follows:

**Definition 2.1.** A fuzzy number u in parametric form is a pair $(u, \overline{u})$ of functions $u(r), \overline{u}(r), 0 \leq r \leq 1$, which satisfy the following requirements:

1. $u(r)$ is a bounded non-decreasing left continuous function in $(0,1]$, and right continuous at 0,
2. $\overline{u}(r)$ is a bounded non-increasing left continuous function in $(0,1]$, and right continuous at 0,
3. $u(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

The trapezoidal fuzzy number $u = (x_0, y_0, \sigma, \beta)$, with two defuzzifiers $x_0, y_0$ and left fuzziness $\sigma > 0$ and right fuzziness $\beta > 0$ is a fuzzy set where the membership function is as follows:

$$u(x) = \begin{cases} \frac{1}{\sigma}(x - x_0 + \sigma) & x_0 - \sigma \leq x \leq x_0, \\ 1 & x_0 \leq x \leq y_0, \\ \frac{1}{\beta}(y_0 - x + \beta) & y_0 \leq x \leq y_0 + \beta, \\ 0 & \text{otherwise}. \end{cases}$$

if $x_0 = y_0$, then $u$ is called a triangular fuzzy number and we write $u = (x_0, \sigma, \beta)$. The support for fuzzy number $u$ is defined as follows: $\sup u = \{x | u(x) > 0\}$, where $\{x | u(x) > 0\}$ is the closure of the set $\{x | u(x) > 0\}$. For arbitrary fuzzy numbers $\overline{u}(r) = [u(r), \overline{u}(r)]$ and $\underline{u}(r) = [\underline{u}(r), \overline{u}(r)]$, we shall define addition, subtraction and multiplication as follows for $0 \leq r \leq 1$:

1. Addition: $(\overline{u} + \overline{v})(r) = \overline{u}(r) + \overline{v}(r), \underline{u}(r), \underline{v}(r)]$
2. Subtraction: $(\overline{u} - \overline{v})(r) = [\underline{u}(r) - \overline{v}(r), \underline{u}(r), \underline{v}(r)]$
3. Multiplication: $(\overline{u} \cdot v)(r) = [\min(\underline{u}(r) v(r), \underline{u}(r) \overline{v}(r), \underline{v}(r) \overline{u}(r), \overline{v}(r) \overline{v}(r))], \max(\underline{u}(r) v(r), \underline{u}(r) \overline{v}(r), \underline{v}(r) \overline{u}(r), \overline{v}(r) \overline{v}(r))].$

**Definition 2.2.** An $m$-set of BPFs is defined over a real interval $[0,H)$ as $[5,14,15]$

$$\Phi_i(t) = \begin{cases} 1, & \frac{iH}{m} \leq t \leq \frac{(i+1)H}{m}, \\ 0, & \text{otherwise}, \end{cases} \quad (2.1)$$

Where $i=0,1,\ldots,m-1$, with a positive integer value for $m$. also consider $h=H/m$, and $\Phi_i$ is the $i$th BPF. Here, we assume that $H=1$, so BPFs are defined over $[0,1)$, and $h=1/m$. a set of BPFs over $[0,1)$ for $m=4$ is shown in fig 1.

There are some properties for BPFs, the most important properties are disjointness, orthogonality, and completeness. Let us consider the first $m$ terms of BPFs and write them concisely as an $m$-vector

$$\phi(t) = [\phi_0(t), \phi_1(t), \ldots, \phi_{m-1}(t)]^T, \quad t \in [0,1) \quad (2.2)$$

Where superscript $T$ indicates transposition. The above representation and disjointness property follows.
\begin{align}
&\phi(t)\phi^T(t)V = \tilde{V}\phi(t) \quad (2.3) \\
\text{Where } V \text{ is an } m\text{-vector and } \tilde{V} = \text{diag}(V). \text{ moreover it can be clearly concluded that for any } m \times m \text{ matrix } B \\
&\phi(t)B\phi^T(t)V = B^T\phi(t) \quad (2.4) \\
\text{Where } \hat{B} \text{ is an } m\text{-vector with elements equal to the diagonal entries of matrix } B. \text{ also} \\
&\int_0^1 \phi(t) \, dt = [h \ h \ \cdots \ h]^T = \tilde{h}, \quad (2.5) \\
\text{And} \\
&\int_0^1 \phi(t) \phi(t)^T \, dt = hI, \quad (2.6) \\
\text{Where } I \text{ is } m \times m \text{ identity matrix.}
\end{align}

**Definition 2.3.** (BPFs expansion) The expansion of a function \( f \) over \([0,1)\) with respect to \( \varphi_i, i=0,1,\ldots,m-1 \), may be compactly written as \([5,14,15]\)

\[
f(t) \approx \sum_{i=0}^{m-1} f_i \varphi_i(t) = F^T\phi(t) = \phi(t)^T F, \quad (2.7)
\]

Where \( F = [f_0 \ f_1 \ \cdots \ f_{m-1}]^T \) and \( f_i \) are defined by
\[ f_i = \frac{1}{h} \int_0^1 f(t) \varphi_i(t) \, dt. \]  
\hspace{1cm} (2.8)

**Definition 2.4.** (operational matrix) computing \( \int_0^t \varphi_i(\tau) \, d\tau \) follows \([5,14,15]\) 

\[
\int_0^t \varphi_i(\tau) \, d\tau = \begin{cases} 
0, & t < ih, \\
-ih, & ih \leq t < (i+1)h, \\
h, & (i+1)h \leq t < 1,
\end{cases}
\] \hspace{1cm} (2.9)

*Note that \( t-ih \) equals to \( h/2 \) at mid-point of \([ih, (i+1)h]\). So we can approximate \( t-ih \), for \( ih \leq t < (i+1)h \) by \( h/2 \).*

Now expressing \( \int_0^t \varphi_i(\tau) \, d\tau \) in terms of BPFs gives 

\[
\int_0^t \varphi_i(\tau) \, d\tau \approx \begin{bmatrix} 0 & \cdots & 0 & h/2 & h & \cdots & h \end{bmatrix} \phi(t), \] \hspace{1cm} (2.10)

*In which \( h/2 \) is \( i \)-th component. Therefore, 

\[
\int_0^t \varphi_i(\tau) \, d\tau \approx P \phi(t), \] \hspace{1cm} (2.11)

*Where \( P_{m \times m} \) is called operational matrix of integration and can be represented as 

\[
P = \frac{h}{2} \begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\
0 & 1 & 2 & \cdots & 2 \\
0 & 0 & 1 & \cdots & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \end{bmatrix} \] \hspace{1cm} (2.12)

**Definition 2.5.** Let \( A, B \in \mathbb{R}_f \). If there exists \( C \in \mathbb{R}_f \) such that \( A = B + C \), then \( C \) is called the Hukuhara difference of \( A \) and \( B \), denoted by \( A \oplus B \).

**Definition 2.6.** The generalized Hukuhara difference of two fuzzy numbers \( A, B \in \mathbb{R}_f \) is defined as follows:

\[
A \ominus_{gh} B \equiv C \iff \begin{cases} 
(i) & A = B + C, \\
(ii) & B = A + (-1)C.
\end{cases}
\]

*3 The error bound by BPFs.*

[17]. In this section first of all the 1-cut form of the equations are introduced as follow:

Considering Fig. (2), the representation error (or the residual error) when a differentiable function \( f_i(t) \) is represented in a series of BPFs over every subinterval \( \left[ \frac{i}{m}, \frac{i+1}{m} \right) \)
\[ e_i(t) = f \varphi_i(t) - f(t), \quad t \in \left[ \frac{i}{m}, \frac{i+1}{m} \right) \]  

(3.13)

It can be shown that

\[ \|e\|^2 = \frac{1}{12m^2} \left[ f'(\xi_i) \right]^2, \quad \xi_i \in \left( \frac{i}{m}, \frac{i+1}{m} \right) \]  

(3.14)

It is obvious that the error in BPFs expansion of \( f(t) \) may be written as

\[ e(t) = \sum_{i=0}^{m-1} f_i(t) - f(t), \quad t \in (0, 1) \]  

(3.15)

So

\[ \|e(t)\|^2 = \int_0^1 \left( \sum_{i=0}^{m-1} f_i(t) - f(t) \right)^2 dt. \]  

(3.16)

Now suppose that \( f(t) \) is decomposed to \( m \) functions \( f_i(t) \), defined in the subintervals, such that

\[ f_i(t) = \begin{cases} f(t), & t \in \left( \frac{i}{m}, \frac{i+1}{m} \right) \\ 0, & \text{otherwise} \end{cases}, \quad i = 0, 1, \ldots, m-1. \]  

(3.17)

So we can write

\[ f(t) = f_0(t) + f_1(t) + \cdots + f_{m-1}(t) = \sum_{i=0}^{m-1} f_i(t), \quad t \in (0, 1) \]  

(3.18)

Therefore from (3.16) we obtain

\[ \|e(t)\|^2 = \int_0^1 \left( \sum_{i=0}^{m-1} f_i(t) - f(t) \right)^2 dt. \]  

(3.19)

Or

\[ \|e(t)\|^2 = \sum_{j=0}^{j+1} \left( \sum_{i=j}^{j+1} f_i - f(t) \right)^2 dt \]  

(3.20)
Note that
\[
\begin{cases}
\varphi_j(t) = 0, & t \notin \left[ \frac{j}{m}, \frac{j+1}{m} \right) \\
f_j(t) = 0, & t \notin \left[ \frac{j}{m}, \frac{j+1}{m} \right)
\end{cases}
\] (3.21)

Hence Eq (3.20) is simplified to
\[
\|e(t)\|^2 = \sum_{j=0}^{m-1} \int_{m}^{j+1} |f_j \varphi_j(t) - f_j(t)|^2 dt = \sum_{j=0}^{m-1} \int_{m}^{j+1} |f_j \varphi_j(t) - f_j(t)|^2 dt. 
\] (3.22)

So according to (3.13) we obtain
\[
\|e(t)\|^2 = \sum_{j=0}^{m-1} \int_{m}^{j+1} |e_j|^2 dt 
\] (3.23)

Or
\[
\|e(t)\|^2 = \sum_{j=0}^{m-1} |e_j|^2 , 
\] (3.24)

Which result according to (3.14), in
\[
\|e(t)\|^2 = \sum_{j=0}^{m-1} \frac{1}{12m^3} \left[ f'(\xi_j) \right]^2, \quad \xi_j \in \left[ \frac{j}{m}, \frac{j+1}{m} \right) 
\] (3.25)

Now we note that for all \(j=0,1,\ldots,m-1\), we have
\[
|f'(\xi_j)|^2 \leq \sup |f'(t)|, \quad t \in [0,1) 
\] (3.26)

That gives
\[
|f'(\xi_j)|^2 = \left[ f'(\xi_j) \right]^2 \leq \left( \sup |f'(t)| \right)^2 \quad t \in [0,1) 
\] (3.27)

From (3.25) and (3.27) it is concluded that
\[
\|e(t)\|^2 \leq \sum_{j=0}^{m-1} \frac{1}{12m^3} \left( \sup |f'(t)| \right)^2, 
\] (3.28)

Or
\[
\|e(t)\|^2 \leq m \frac{1}{12m^3} \left( \sup |f'(t)| \right)^2, 
\] (3.29)

And consequently
\[
\|e(t)\|^2 \leq \frac{1}{12m^2} \left( \sup |f'(t)| \right)^2, 
\] (3.30)

Which result in
\[
\|e(t)\| \leq \frac{1}{2\sqrt{3m}} \sup |f'(t)|, 
\] (3.31)

Or
\[
\|e(t)\| \leq M \frac{1}{m}, 
\] (3.32)
Where $M = \frac{1}{2\sqrt{3m}} \lim_{m \to \infty} \sup_{t} |f'(t)|$. Eq. (3.32) clearly shows that the total error in approximation by an m-set of BPFs is $g(\frac{1}{m})$, in other words, the order of convergence may be considered as $g(\frac{1}{m})$. Moreover we obtain from (3.32) that.

$$\lim_{m \to \infty} \|e(t)\| \leq M \lim_{m \to \infty} \left(\frac{1}{m}\right).$$

(3.33)

Therefore

$$\lim_{m \to \infty} \|e(t)\| = 0,$$

(3.34)

Which establishes that we will have an exact representation of the function by using BPFs if m is high enough. Now we are going to allocate the spreads to eq (3.13) to continue the process for finding the error bound. The fig (3) is a representation of the error for (3.13) in fuzzy case.

\[
\begin{align*}
[e_i(t,r) - \alpha(t,r), e_i(t,r) + \alpha(t,r)] &= \\
&= \left[\left(f_i(t,r) - \alpha(t,r), f_i(t,r) + \alpha(t,r)\right)\phi_i(t) - \left(f(t,r) - \alpha(t,r), f(t,r) + \alpha(t,r)\right)\right]
\end{align*}
\]

(3.35)

So that

\[
\begin{align*}
[e_i(t,r) - \alpha(t,r)] &= \left[f_i(t,r) - \alpha(t,r)\right]\phi_i(t) - \left[f(t,r) - \alpha(t,r)\right] \\
[e_i(t,r) + \alpha(t,r)] &= \left[f_i(t,r) + \alpha(t,r)\right]\phi_i(t) - \left[f(t,r) + \alpha(t,r)\right]
\end{align*}
\]

(3.36)

It can be shown that

\[
\begin{align*}
\|e_i(t,r) - \alpha(t,r)\|^2 &= \frac{1}{12m^3}\left[f'\left(\xi_i\right) - \alpha'(t,r)\right]^2 \\
\|e_i(t,r) + \alpha(t,r)\|^2 &= \frac{1}{12m^3}\left[f'\left(\xi_i\right) + \alpha'(t,r)\right]^2
\end{align*}
\]

(3.37)

It is obvious that the error in the error in the BPFs of $f(t,r)$ may be written as

\[
\begin{align*}
[e(t,r) - \alpha(t,r), e(t,r) + \alpha(t,r)] &= \\
&= \sum_{i=0}^{m-1}\left[\left(f_i(t,r) - \alpha(t,r), f_i(t,r) + \alpha(t,r)\right)\phi_i(t) - \left(f(t,r) - \alpha(t,r), f(t,r) + \alpha(t,r)\right)\right]
\end{align*}
\]

(3.38)

So that

\[
\begin{align*}
[e(t,r) - \alpha(t,r)] &= \sum_{i=0}^{m-1}\left[f_i(t,r) - \alpha(t,r)\right]\phi_i(t) - \left[f(t,r) - \alpha(t,r)\right] \\
[e(t,r) + \alpha(t,r)] &= \sum_{i=0}^{m-1}\left[f_i(t,r) + \alpha(t,r)\right]\phi_i(t) - \left[f(t,r) + \alpha(t,r)\right]
\end{align*}
\]

(3.39)

So

\[
\begin{align*}
\|e(t,r) - \alpha(t,r)\|^2 &= \int_{0}^{1}\left\|\sum_{i=0}^{m-1}\left[f_i(t,r) - \alpha(t,r)\right]\phi_i(t) - \left[f(t,r) - \alpha(t,r)\right]\right\|^2 dt \\
\|e(t,r) + \alpha(t,r)\|^2 &= \int_{0}^{1}\left\|\sum_{i=0}^{m-1}\left[f_i(t,r) + \alpha(t,r)\right]\phi_i(t) - \left[f(t,r) + \alpha(t,r)\right]\right\|^2 dt, \forall r \in [0,1]
\end{align*}
\]

(3.40)

Now suppose that $f(t,r)$ is decomposed to m functions $f_i(t,r)$, defined in the subinterval, such that
\begin{align}
f_i(t, r) &= \begin{cases} f(t, r), & t \in \left[ \frac{i}{m}, \frac{i+1}{m} \right) \\ 0, & \text{otherwise} \end{cases} \quad (3.41)
\end{align}

so we can write

\begin{align}
f(t, r) &= f_0(t, r) + f_1(t, r) + \cdots + f_{m-1}(t, r) = \sum_{i=0}^{m-1} f_i(t, r), \quad \forall \ t \in (0,1), \ \forall \ r \in [0,1] \quad (3.42)
\end{align}

Therefore from (3.40) we obtain

\begin{align}
\|e(t, r) - \alpha(t, r)\|^2 &= \int_{0}^{1} \left\| \sum_{i=0}^{m-1} \left[ f_i(t, r) - \alpha(t, r) \right] \varphi_i(t) - \left[ f(t, r) - \alpha(t, r) \right] \right\|^2 dt, \\
\|e(t, r) + \alpha(t, r)\|^2 &= \int_{0}^{1} \left\| \sum_{i=0}^{m-1} \left[ f_i(t, r) + \alpha(t, r) \right] \varphi_i(t) - \left[ f(t, r) + \alpha(t, r) \right] \right\|^2 dt
\end{align}

or

\begin{align}
\|e(t, r) - \alpha(t, r)\|^2 &= \sum_{j=0}^{j+1} \left[ \sum_{i=0}^{m-1} \left[ f_i(t, r) - \alpha(t, r) \right] \varphi_i(t) - \left[ f(t, r) - \alpha(t, r) \right] \right]^2 dt \\
\|e(t, r) + \alpha(t, r)\|^2 &= \sum_{j=0}^{j+1} \left[ \sum_{i=0}^{m-1} \left[ f_i(t, r) + \alpha(t, r) \right] \varphi_i(t) - \left[ f(t, r) + \alpha(t, r) \right] \right]^2 dt \quad (3.45)
\end{align}

Note that

\begin{align}
\varphi_j(t) &= 0, \\
f_j(t, r) &= 0, \quad t \notin \left[ \frac{j}{m}, \frac{j+1}{m} \right) \quad (3.46)
\end{align}

Hence Eq (3.45) is simplified to

\begin{align}
\|e(t, r) - \alpha(t, r)\|^2 &= \sum_{j=0}^{j+1} \left[ \sum_{i=0}^{m-1} \left[ f_j(t, r) - \alpha_j(t, r) \right] \varphi_j(t) - \left[ f(t, r) - \alpha_j(t, r) \right] \right]^2 dt \\
&= \sum_{j=0}^{j+1} \left[ \sum_{i=0}^{m-1} \left[ f_j(t, r) - \alpha_j(t, r) \right] \varphi_j(t) - \left[ f(t, r) - \alpha_j(t, r) \right] \right]^2 dt \quad (3.47)
\end{align}

\begin{align}
\|e(t, r) + \alpha(t, r)\|^2 &= \sum_{j=0}^{j+1} \left[ \sum_{i=0}^{m-1} \left[ f_j(t, r) + \alpha_j(t, r) \right] \varphi_j(t) - \left[ f(t, r) + \alpha_j(t, r) \right] \right]^2 dt \\
&= \sum_{j=0}^{j+1} \left[ \sum_{i=0}^{m-1} \left[ f_j(t, r) + \alpha_j(t, r) \right] \varphi_j(t) - \left[ f(t, r) + \alpha_j(t, r) \right] \right]^2 dt \quad (3.48)
\end{align}
so according to (3.35) we obtain
\[
\|e(t,r) - \alpha(t,r)\|^2 = \sum_{j=0}^{m-1} \left( \int_{\frac{j}{m}}^{\frac{j+1}{m}} \left| e_j(t,r) - \alpha_j(t,r) \right|^2 dt \right),
\]
(3.49)
\[
\|e(t,r) + \alpha(t,r)\|^2 = \sum_{j=0}^{m-1} \left( \int_{\frac{j}{m}}^{\frac{j+1}{m}} \left| e_j(t,r) + \alpha_j(t,r) \right|^2 dt \right),
\]
(3.50)
Or
\[
\|e(t,r) - \alpha(t,r)\|^2 = \sum_{j=0}^{m-1} \left( \int_{\frac{j}{m}}^{\frac{j+1}{m}} \left| e_j(t,r) - \alpha_j(t,r) \right|^2 dt \right)
\]
\[
\|e(t,r) + \alpha(t,r)\|^2 = \sum_{j=0}^{m-1} \left( \int_{\frac{j}{m}}^{\frac{j+1}{m}} \left| e_j(t,r) + \alpha_j(t,r) \right|^2 dt \right)
\]
(3.51)
Which result according to (3.36), in
\[
\|e(t,r) - \alpha(t,r)\|^2 = \sum_{j=0}^{m-1} \frac{1}{12m^3} \left[f'(\xi_j) - \alpha'(t,r)\right]^2
\]
\[
\|e(t,r) + \alpha(t,r)\|^2 = \sum_{j=0}^{m-1} \frac{1}{12m^3} \left[f'(\xi_j) + \alpha'(t,r)\right]^2
\]
\[
\xi_j \in \left[\frac{j}{m}, \frac{j+1}{m}\right)
\]
(3.52)
\[
\|e(t,r) - \alpha(t,r)\|^2 = \sum_{j=0}^{m-1} \left( \int_{\frac{j}{m}}^{\frac{j+1}{m}} \left| f'(\xi_j) - \alpha(t,r) \right| dt \right)
\]
\[
\|e(t,r) + \alpha(t,r)\|^2 = \sum_{j=0}^{m-1} \left( \int_{\frac{j}{m}}^{\frac{j+1}{m}} \left| f'(\xi_j) + \alpha(t,r) \right| dt \right)
\]
\[
\xi_j \in \left[\frac{j}{m}, \frac{j+1}{m}\right)
\]
(3.53)
Now, we note that for all $j=0,1,\ldots,m-1$, we have
\[
\left| f'(\xi_j, r) - \alpha(t,r) \right| \leq \sup_{t \in (0,1)} |f'(t,r) - \alpha(t,r)|
\]
\[
\left| f'(\xi_j, r) + \alpha(t,r) \right| \leq \sup_{t \in (0,1)} |f'(t,r) + \alpha(t,r)|
\]
(3.54)
that gives
\[
\exists \sup_{t \in (0,1)} |\alpha(t,r)| \leq m'
\]
\[
\leq \sup_{t \in (0,1)} |f'(t,r)| + m'
\]
\[
= M + m'
\]
(3.55)
Then suppose $\sup_{t \in (0,1)} |f'(t,r)| = M$
\[
\left| f'(\xi_j) - \alpha(t,r) \right|^2 \leq \sup_{t \in (0,1)} |f'(t,r) - \alpha(t,r)|^2
\]
\[
\left| f'(\xi_j) + \alpha(t,r) \right|^2 \leq \sup_{t \in (0,1)} |f'(t,r) + \alpha(t,r)|^2
\]
(3.56)
From (3.53) and (3.55) it is concluded that
\[
\left\| e(t,r) - \alpha(t,r) \right\|^2 \leq \sum_{j=0}^{m-1} \frac{1}{12m^3} \left[ \sup |f'(t,r) - \alpha'(t,r)| \right]^2,
\]
\[
= \sum_{j=0}^{m-1} \frac{1}{12m^3} [M+m']^2,
\]
\[
\left\| e(t,r) + \alpha(t,r) \right\|^2 \leq \sum_{j=0}^{m-1} \frac{1}{12m^3} \left[ \sup |f'(t,r) + \alpha'(t,r)| \right]^2
\]
\[
= \sum_{j=0}^{m-1} \frac{1}{12m^3} [M+m']^2
\]
(3.56)

Or
\[
\left\| e(t,r) - \alpha(t,r) \right\|^2 \leq m \cdot \frac{1}{12m^3} [[M+m']]^2,
\]
\[
\left\| e(t,r) + \alpha(t,r) \right\|^2 \leq m \cdot \frac{1}{12m^3} [M+m']^2
\]
(3.57)

And consequently
\[
\left\| e(t,r) - \alpha(t,r) \right\|^2 \leq \frac{1}{12m^3} \left[ \sup |f'(t,r) - \alpha'(t,r)| \right]^2,
\]
\[
= \frac{1}{12m^3} [M+m']^2,
\]
\[
\left\| e(t,r) + \alpha(t,r) \right\|^2 \leq \frac{1}{12m^3} \left[ \sup |f'(t,r) + \alpha'(t,r)| \right]^2
\]
\[
\leq \frac{1}{12m^3} [M+m']^2
\]
(3.58)

Which result in
\[
\left\| e(t,r) - \alpha(t,r) \right\| \leq \frac{1}{2\sqrt{3m}} \left[ \sup |f'(t,r) - \alpha'(t,r)| \right]
\]
\[
= \frac{1}{2\sqrt{3m}} [M+m']
\]
\[
\left\| e(t,r) + \alpha(t,r) \right\| \leq \frac{1}{2\sqrt{3m}} \left[ \sup |f'(t,r) + \alpha'(t,r)| \right]
\]
\[
= \frac{1}{2\sqrt{3m}} [M+m']
\]
(3.59)

Or
\[
\left\| e(t,r) - \alpha(t,r) \right\| \leq M \cdot \frac{1}{m},
\]
\[
\left\| e(t,r) + \alpha(t,r) \right\| \leq M \cdot \frac{1}{m}
\]
(3.60)

Where \( M = \frac{1}{2\sqrt{3m}} \sup |f'(t,r) - \alpha'(t,r)| \) or \( M = \frac{1}{2\sqrt{3m}} \sup |f'(t,r) + \alpha'(t,r)| \).
Eq. (3.60) clearly shows that the total error in approximation by an \( m \)-set of BPFs is \( \Theta \left( \frac{1}{m} \right) \), in other words, the order of convergence may be considered as \( \Theta \left( \frac{1}{m} \right) \). Moreover we obtain from (3.60) that.

\[
\lim_{m \to \infty} \|e(t,r) - \alpha(t,r)\| \leq M \lim_{m \to \infty} \left( \frac{1}{m} \right), \\
\lim_{m \to \infty} \|e(t,r) + \alpha(t,r)\| \leq M \lim_{m \to \infty} \left( \frac{1}{m} \right)
\]  

(3.61)

Therefore

\[
\lim_{m \to \infty} \|e(t,r) - \alpha(t,r)\| = 0, \\
\lim_{m \to \infty} \|e(t,r) + \alpha(t,r)\| = 0
\]  

(3.62)

Which establishes that we will have an exact representation of the function by using BPFs if \( m \) is high enough.

4 Conclusion

It is clear that, the error analysis of fuzzy models and fuzzy methods is different from crisp case. Using the mentioned method in this paper, the error and its upper bound are analyzed and investigated.

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