Approximation solution of the complex modified Korteweg-de Vries equation via the homotopy analysis method

S. Naghshband\textsuperscript{1}, M. A. Fariborzi Araghi\textsuperscript{2}\textsuperscript{*}

\textsuperscript{(1)} Department of Mathematics, West Tehran Branch, Islamic Azad University, Tehran, Iran.
\textsuperscript{(2)} Department of Mathematics, Central Tehran Branch, Islamic Azad University, Zip code 14676-86831, Tehran, Iran.

Copyright 2017 © S. Naghshband and M. A. Fariborzi Araghi. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract
In this paper, the homotopy analysis method (HAM) is used to obtain the series solution of the complex modified Korteweg-de Vries (CMKDV) equation. The HAM is compared with some other methods, also a convergence theorem is proved. Two sample CMKDV equations are solved to demonstrate the efficiency of the suggested technique. By plotting the $h$-curve of the examples, the region of convergence is determined.

Keywords: Homotopy Analysis Method (HAM), Convergence, Nonlinearity, Complex Modified Korteweg-de Vries (CMKDV) equation.

1 Introduction
Let the complex modified Korteweg-de Vries (CMKDV) equation as follows:
\[ w_t + w_{xxx} + \alpha(|w|^2 w)_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \]  
(1.1)
where $w$ is a complex valued function of the spatial coordinate of $x$ and the time coordinate of $t$, also $\alpha$ is a real parameter [22,24,26]. Eq. (1.1) has been suggested as a model for the nonlinear evolution of plasma waves. Eq. (1.1) includes the propagation of transverse waves in a molecular chain model [13]. A solitary wave solution for the Eq. (1.1) can be considered as

$$w(x,t) = \sqrt{2c/\alpha} \text{sech}(\sqrt{c}(x-x_0-ct)) \exp(i\theta_0), t^2 = -1.$$ (1.2)

This single solitary wave of amplitude $\sqrt{2c/\alpha}$ is centered at $x_0$ and going to the right with velocity $c$ and satisfies the boundary conditions $w \to 0$ when $x \to \pm \infty$ [22], also in two discussed examples $\theta_0$ is considered 0 and $\pi/4$ respectively. Ismail solved the CMKdV equation by collocation method and Petrov-Galerkin method [12, 13]. Korkmaz and Dag solved the CMKdV equation by means of differential quadrature method based on cosine expansion [14]. Muslu and Erbay solved the CMKdV equation via three different split-step Fourier schemes [19]. Salkuyeh and Bastani solved Eq. (1.1) by the projected differential transform method (PDTM) [22]. Hizel gives a classification of group invariant solutions for the CMKdV equation by applying classical Lie method [10]. Uddin, Haq and Islam solved CMKDV equation numerically by mesh-free collocation method [24]. Wazwaz solved CMKDV equation by means of the tanh method and the sine-cosine method [25]. In this work, the HAM is applied to solve the Eq. (1.1). In recent years, this method as an approximate-analytical scheme was used to solve different kinds of problems in science and engineering [1-9, 11, 15, 17, 18, 20, 21, 23]. The homotopy analysis method contains an auxiliary parameter $h$ which provides us a simple way to adjust and control the convergent region of the series solution. The HAM bears a very rapid convergence of the solution series in most cases, usually only a few iterations with satisfactory approximate solution, as well. Total discription of this paper is as follows: In section 2, some preliminaries are given, and in section 3, the main idea of this paper is explained, in section 4 convergence theorem of the HAM is proved, and finally in section 5, two CMKDV equations are solved by the proposed method and compared with mentioned method in [22], and $h$-curves are plotted to show the region of convergence.

2 Preliminaries

Let the nonlinear differential equation as follows:

$$N[w(x,t)] = 0,$$

where $N$ is a nonlinear operator, $x$ and $t$ denote the independent variables and $w$ is an unknown function. In the HAM, the zeroth-order deformation equations is defined as:

$$(1-q)L[\Phi(x,t;q) - w_0(x,t)] = qhH(x,t)N[\Phi(x,t;q)],$$ (2.3)

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is an auxiliary parameter, $L$ is an auxiliary linear operator and $H(x,t)$ is an auxiliary function. $\Phi(x,t;q)$ is an unknown function and $w_0(x,t)$ is an initial guess of $w(x,t)$. It is obvious, if $q = 0$ and $q = 1$ then:

$$\Phi(x,t;0) = w_0(x,t), \quad \Phi(x,t;1) = w(x,t),$$

respectively. Therefore, when $q$ increases from 0 to 1, the solution $\Phi(x,t;q)$ varies from $w_0(x,t)$ to the exact solution $w(x,t)$. By Taylor’s theorem, it can be expanded $\Phi(x,t;q)$ in a power series of the embedding parameter $q$ as:

$$\Phi(x,t;q) = w_0(x,t) + \sum_{m=1}^{\infty} w_m(x,t)q^m,$$ (2.4)

where
\[
\frac{\partial w_m(x,t)}{\partial t} + \frac{\partial^3 w_m(x,t)}{\partial x^3} + \alpha \frac{\partial (w^3 w_m(x,t))}{\partial x} = 0, \quad w(x,0) = f(x),
\]
(3.11)

and
\[
L[\Phi(x,t; q)] = \frac{\partial \Phi(x,t; q)}{\partial t}, \quad L(c) = 0,
\]
(3.12)

where \( c \) is a real constant,
\[
N[\Phi(x,t; q)] = \frac{\partial \Phi(x,t; q)}{\partial t} + \frac{\partial^3 \Phi(x,t; q)}{\partial x^3} + \alpha \frac{\partial (\Phi^3(x,t; q)\Phi(x,t; q))}{\partial x},
\]
(3.13)

and \( H(x,t) = 1 \). The zeroth-order and the \( m \)-th order deformation equations are:
\[
(1-q)L[\Phi(x,t; q) - w_0] = qhN[\Phi(x,t; q)].
\]
(3.14)
\[
L[w_m - \chi_m w_{m-1}] = hR_m(\tilde{w}_{m-1}),
\]
(3.15)

where
\[
R_m(\tilde{w}_{m-1}) = \frac{\partial w_{m-1}}{\partial t} + \frac{\partial^3 w_{m-1}}{\partial x^3} + \alpha \frac{\partial \left( \sum_{k=0}^{m-1} \sum_{j=0}^{m-1-k} w_j w_{m-1-k-j} \tilde{w}_k \right)}{\partial x}.
\]
(3.16)
So,
\[ w_m = \chi_m w_{m-1} + h \int_0^w R_m (w_{m-1}) \, dt + c, \quad m \geq 1. \tag{3.17} \]

4 Convergence of the HAM

In this part, the convergence of the series solution obtained from the HAM to the exact solution of the Eq. (3.11) is proved.

**Theorem 4.1.** If the series solution
\[ w(x, t) = w_0(x, t) + w_1(x, t) + \ldots, \]
obtained from the HAM and \( \sum_{m=0}^{\infty} \frac{\partial w_m}{\partial t}, \sum_{m=0}^{\infty} \frac{\partial^3 w_m}{\partial x^3} \) are convergent, then \( \sum_{m=0}^{\infty} w_m \) converges to the exact solution of the Eq. (3.11).

**proof.** Let the series
\[ \sum_{m=0}^{\infty} w_m(x, t) \]
be convergent. Then we can write
\[ w(x, t) = \sum_{m=0}^{\infty} w_m(x, t). \]
In this case,
\[ \lim_{m \to \infty} w_m(x, t) = 0. \tag{4.18} \]

So
\[ \sum_{m=1}^{\infty} [w_m(x, t) - \chi_m w_{m-1}(x, t)] = w_0(x, t) \tag{4.19} \]

By using Eq. (4.18),
\[ \sum_{m=1}^{\infty} [w_m(x, t) - \chi_m w_{m-1}(x, t)] \rightarrow \lim_{n \to \infty} w_n(x, t) = 0, \tag{4.20} \]
therefore,
\[ \sum_{m=1}^{\infty} L[w_m(x, t) - \chi_m w_{m-1}(x, t)] = L\left[ \sum_{m=1}^{\infty} (w_m(x, t) - \chi_m w_{m-1}(x, t)) \right] = 0. \tag{4.21} \]

By applying
\[ L[w_m(x, t) - \chi_m w_{m-1}] = hH(x, t) R_m (w_{m-1}), \tag{4.22} \]
and
\[ \sum_{m=1}^{\infty} L[w_m(x, t) - \chi_m w_{m-1}] = hH(x, t) \sum_{m=1}^{\infty} R_m (w_{m-1}), \tag{4.23} \]
also, since \( h, H(x, t) \neq 0 \) and from Eq. (4.21) we get:
\[ \sum_{m=1}^{\infty} [R_m (w_{m-1})] = 0. \tag{4.24} \]
According to Eq. (3.16),
\[ \sum_{m=1}^{\infty} R_m (w_{m-1}) = \sum_{m=1}^{\infty} \frac{\partial w_{m-1}}{\partial t} + \sum_{m=1}^{\infty} \frac{\partial^3 w_{m-1}}{\partial x^3} + \alpha \frac{\partial \sum_{m=1}^{\infty} \sum_{j=0}^{m} \sum_{k=0}^{\infty} w_j w_{m-1-k-j} w_k}{\partial x} = \]
\[ \sum_{m=0}^{\infty} \frac{\partial w_m}{\partial t} + \sum_{m=0}^{\infty} \frac{\partial^3 w_m}{\partial x^3} + \alpha \frac{\partial \sum_{m=0}^{\infty} \sum_{j=0}^{m} \sum_{k=0}^{\infty} w_j w_{m-1-k-j} w_k}{\partial x} = \]
\[ \sum_{m=0}^{\infty} \frac{\partial w_m}{\partial t} + \sum_{m=0}^{\infty} \frac{\partial^3 w_m}{\partial x^3} + \alpha \frac{\partial \sum_{m=0}^{\infty} \sum_{j=0}^{m} \sum_{k=0}^{m-1} w_j w_{m-1-j} w_k}{\partial x} = \]
\[ \sum_{m=0}^{\infty} \frac{\partial w_m}{\partial t} + \sum_{m=0}^{\infty} \frac{\partial^3 w_m}{\partial x^3} + \alpha \frac{\partial \sum_{m=0}^{\infty} \sum_{j=0}^{m} \sum_{k=0}^{m-1} w_j w_{m-1-j} w_k}{\partial x} = \]
\[ \sum_{m=0}^{\infty} \frac{\partial w_m}{\partial t} + \sum_{m=0}^{\infty} \frac{\partial^3 w_m}{\partial x^3} + \alpha \frac{\partial \sum_{m=0}^{\infty} \sum_{j=0}^{m} \sum_{k=0}^{m} w_j w_{m-1-j} w_k}{\partial x} = \]
\[ \sum_{m=0}^{\infty} \frac{\partial w_m}{\partial t} + \sum_{m=0}^{\infty} \frac{\partial^3 w_m}{\partial x^3} + \alpha \frac{\partial \sum_{m=0}^{\infty} \sum_{j=0}^{m} \sum_{k=0}^{m} w_j w_{m-1-j} w_k}{\partial x} = \]
\[ \sum_{m=0}^{\infty} \frac{\partial w_m}{\partial t} + \sum_{m=0}^{\infty} \frac{\partial^3 w_m}{\partial x^3} + \alpha \frac{\partial \sum_{m=0}^{\infty} \sum_{j=0}^{m} \sum_{k=0}^{m} w_j w_{m-1-j} w_k}{\partial x} = \]
\[ \frac{\partial \sum_{m=0}^{\infty} \sum_{j=0}^{m} \sum_{k=0}^{m} w_j w_{m-1-j} w_k}{\partial t} + \alpha \frac{\partial \sum_{m=0}^{\infty} \sum_{j=0}^{m} \sum_{k=0}^{m} w_j w_{m-1-j} w_k}{\partial x} = \]

Hence, from Eqs. (4.24) and (4.25), it can be concluded that \( w(x,t) = \sum_{m=0}^{\infty} w_m(x,t) \) is the exact solution of Eq. (3.11) and the proof is completed.

5 Sample examples

In this section, two CMKDV equations are solved by the HAM. Also, the region of convergence is shown in the HAM by plotting the \( h \)-curves. The programs have been provided and the figures have been plotted by Matlab package.
Example 5.1. Let the following CMKDV equation:
\[ w_t + w_{xxx} + 2|w|^2 w_x = 0, \ w(x,0) = sech(x-15)[14,22]. \]
We solve the equation by the HAM, using Eq.(3.16), we get:
\[ w_0(x,t) = 1/cosh(x-15), \]
\[ w_1(x,t) = -(htsinh(x-15))/(sinh(x-15)^2 +1), \]
\[ w_2(x,t) = -(3h^2t^2 + 2htsinh(2x-30) - h^2t^2cosh(2x-30) + 2htsinh(2x-30))((4cosh(x-15))^3), \]
\[ w_3(x,t) = (h^2t^2(h+1))/cosh(x-15) - (2h^2t^2(h+1))/cosh(x-15)^3 + (h^3t^3 sinh(x-15))/cosh(x-15)^4 - (htsinh(x-15)(h^2t^2 + 6h^2 + 12h + 6))/(6cosh(x-15)^2), \ldots \]

Table 1 shows the errors of the HAM for different values of \( t \) and \( h \), when \( x=15(\text{center}) \), it can be seen that proper and different approximations can be achieved, by changing \( h \), also error is calculated by \( \sum_{i=0}^{n} w_i - w \). Fig 1 shows the \( h \)-curve of 4-approximation of example 5.1, when \( x=15 \) and \( t=1 \), which makes it possible to choose the appropriate values of \( h \), in other words, the region of convergence at this point is determined.

<table>
<thead>
<tr>
<th>( t/h )</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>2.2743e-011</td>
<td>8.8901e-008</td>
<td>1.0719e-005</td>
<td>3.0789e-004</td>
<td>4.0152e-003</td>
</tr>
<tr>
<td>-0.8</td>
<td>5.3421e-008</td>
<td>1.0669e-006</td>
<td>1.4425e-005</td>
<td>1.7445e-004</td>
<td>5.5140e-005</td>
</tr>
</tbody>
</table>

Fig 2 shows the \( h \)-curve of 5-approximation of example 5.1 at the point (30,1), therefore in this situation, the convergence region is \(-2 < h < 0\). Table 2 displays the errors of proposed method, when \( x=30 \) and \( n=8 \) for different values of \( t \) and \( h \). Fig 3 compares the numerical solution with the exact solution, when \( x \in [10,20] \), \( t \in [0,1] \) and \( h = -1 \).
Figure 2: The $h$-curve of 5-approximation ($n=5$) of example 5.1 when $x=30$ and $t=1$.

Table 2: The errors of the HAM at $x=30$ and $n=8$, for example 5.1.

<table>
<thead>
<tr>
<th>$t/h$</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>8.8038e-019</td>
<td>4.6031e-016</td>
<td>1.8069e-014</td>
<td>2.4580e-013</td>
<td>1.8713e-012</td>
</tr>
<tr>
<td>-0.8</td>
<td>2.8795e-012</td>
<td>2.3341e-011</td>
<td>1.0403e-010</td>
<td>3.4501e-010</td>
<td>9.5255e-010</td>
</tr>
<tr>
<td>-1.5</td>
<td>7.0006e-011</td>
<td>9.6624e-011</td>
<td>1.9261e-010</td>
<td>3.5011e-010</td>
<td>5.8713e-011</td>
</tr>
</tbody>
</table>

Table 3 compares the results of the HAM with the schemes mentioned in [22] such as differential transformation method (DTM) at $t=0.5$. It can be seen that the HAM is more successful, also for two points (0.5,0.5) and (10,0.5). $h$ has been considered -0.87, and for the point (20, 0.5), $h = -1$.

Table 3: The comparison of the HAM with the other methods at $t=0.5$, for example 5.1.

<table>
<thead>
<tr>
<th>$x$</th>
<th>Exact</th>
<th>VIM, n=4</th>
<th>DTM, n=5</th>
<th>PDTM, n=5</th>
<th>HAM, n=4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>6.118046410e-7</td>
<td>6.120469032e-7</td>
<td>6.118046410e-7</td>
<td>6.117842212e-7</td>
<td>6.118485399e-7</td>
</tr>
<tr>
<td>10</td>
<td>8.173406367e-3</td>
<td>8.159142736e-3</td>
<td>7.235910676e-4</td>
<td>8.173141249e-3</td>
<td>8.173993242e-3</td>
</tr>
<tr>
<td>20</td>
<td>2.221525150e-2</td>
<td>2.223279605e-2</td>
<td>1.894569839e-2</td>
<td>2.221494954e-2</td>
<td>2.2211478323e-2</td>
</tr>
</tbody>
</table>
Figure 3: Numerical solution (left) and exact solution of example 5.1, when \( h = 1 \) and \( n = 6 \).

**Example 5.2.** Consider the following CMKDV equation:

\[
w_t + w_{xx} + 2(|w|^2 w)_x = 0, \quad w(x,0) = \sqrt{2/2(1+i)} \text{sech}(x-15), \quad i^2 = -1[22].
\]

By using Eqs. (3.16) and (3.17), namely HAM, it can be seen:

\[
w_0(x,t) = \left(2^{1/2} (i/2 + 1/2)/\cosh(x-15),
\right.
\]

\[
w_1(x,t) = \left(2^{1/2} \text{htsinh}(x-15)(-i/2 - 1/2))(\sinh(x-15)^2 + 1),
\right.
\]

\[
w_2(x,t) = \left(-2^{1/2} h^2 t^2 ((3i)/2 + 3/2) + 2^{1/2} \text{htsinh}(2x - 30)(i + 1) + 2^{1/2} h^2 \text{tsh}(2x - 30)(i + 1) + 2^{1/2} h^2 \text{th}(2x - 30)(-i/2 - 1/2))(\cosh(x-15)^3),
\right.
\]

\[
w_3(x,t) = \left(2^{1/2} h^3 t^3 \sinh(x-15)(i/2 + 1/2) - (2^{1/2}) \text{htcosh}(x-15)^2 (\sinh(x-15)^2 t^2 (i/6 + 1/6) + \sinh(x-15)h^2 (i + 1) + \sinh(x-15)(2i + 2) + \sinh(x-15)(i + 1))/2 - (2^{1/2}) \text{htcosh}(x-15)(th^2 (2i + 2) + \text{th}(2i + 2))/2 + (21/2) \text{htcosh}(x-15)^3 (th^2 (i + 1) + \text{th}(i + 1))/2)/\cosh(x-15)^3, \ldots
\]

Table 4 compares the HAM with the other mentioned methods in [22], and easily it can be observed that the proposed method can be considered as a reliable scheme, also for two points \((0.5,0.5)\) and \((10,0.5)\), \( h \) has been considered \(-0.9\), and for the point \((20,0.5)\), \( h \) is \(-1.15\).

<table>
<thead>
<tr>
<th>( x )</th>
<th>Exact</th>
<th>VIM, ( n=2 )</th>
<th>DTM, ( n=4 )</th>
<th>PDTM, ( n=4 )</th>
<th>HAM, ( n=2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>4.326112104e-7</td>
<td>4.457845652e-7</td>
<td>4.326112104e-7</td>
<td>4.327825156e-7</td>
<td>4.324110284e-7</td>
</tr>
<tr>
<td>10</td>
<td>5.779471067e-3</td>
<td>5.951558561e-3</td>
<td>2.327008941e-4</td>
<td>5.781737980e-3</td>
<td>5.776708837e-3</td>
</tr>
<tr>
<td>20</td>
<td>1.570855498e-2</td>
<td>1.548783942e-2</td>
<td>1.523048491e-3</td>
<td>1.570588707e-2</td>
<td>1.575967916e-2</td>
</tr>
</tbody>
</table>

Table 5 shows the errors of the HAM of example 5.2, at the point \((10,0.5)\), for different values of \( h \). Fig 4 displays the \( h \)-curve of 4-approximation of example 5.2, when \( x = 10 \), \( t = 0.5 \) and \( n = 4 \), and thus the proper values of \( h \) is determined.
Table 5: The errors of the HAM at $x=10$ and $t=0.5$.

<table>
<thead>
<tr>
<th>$h/n$</th>
<th>-1</th>
<th>-0.9</th>
<th>-0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2.4865e-004</td>
<td>3.9064e-006</td>
<td>8.8041e-005</td>
</tr>
<tr>
<td>4</td>
<td>3.2058e-006</td>
<td>2.9027e-007</td>
<td>1.7603e-006</td>
</tr>
<tr>
<td>6</td>
<td>1.7914e-008</td>
<td>1.9554e-009</td>
<td>4.0845e-008</td>
</tr>
<tr>
<td>8</td>
<td>1.3026e-011</td>
<td>1.6231e-011</td>
<td>2.9264e-009</td>
</tr>
</tbody>
</table>

Figure 4: $h$-curve of $w_n$ of example 5.2, when $x=10$, $t=0.5$ and $n=4$. In this example, it is clear that the imaginary part and the real part of the solutions are the same and then in Fig 5 only the real part of solutions are drawn, this figure displays the similarity of the numerical solution and the exact solution, when $x \in [5,25]$, $t \in [0,1]$ and $h=-1$.

Figure 5: Real part of numerical solution(left) and exact solution of example 5.2, when $h=-1$ and $n=5$. 
6 Conclusion

In this paper, the homotopy analysis method was used to solve the Complex Modified Korteweg-de Vries equation. Also, two examples were solved and the $h$-curves of the examples were drawn and some numerical results were presented to show the importance and applicability of the HAM. Consequently, the HAM can be applied to solve the CMKDV equation as an effective and valid scheme.

References


https://doi.org/10.1016/0010-4655(91)90165-H

https://doi.org/10.1016/j.cnsns.2007.12.005

https://doi.org/10.1016/j.amc.2008.02.033

https://doi.org/10.1016/j.cpc.2009.04.012

https://doi.org/10.1016/S0096-3003(02)00790-7

https://doi.org/10.1201/9780203491164

https://doi.org/10.1016/j.cnsns.2008.04.013


https://doi.org/10.1016/S0898-1221(03)80033-0

https://doi.org/10.1016/j.cnsns.2012.01.032

https://doi.org/10.1007/s11071-006-9140-y
https://doi.org/10.1016/j.amc.2012.11.062

https://doi.org/10.1016/0378-4754(94)00031-X

https://doi.org/10.1016/j.camwa.2009.03.104

https://doi.org/10.1016/j.camwa.2004.08.013

https://doi.org/10.1007/978-3-642-00251-9