An Effective scheme based on quartic B-spline for the solution of Gardner equation and Harry Dym equation

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Abstract
In this paper, numerical solution of both the combined KdV and modified KdV equation (KdV-MKdV) known as Gardner equation and Harry Dym equation are obtained using quartic B-spline as a space approximation combined with finite difference for time discretization. We also study the stability analysis using Von-Neumann stability analysis showing that the numerical scheme is unconditionally stable. Accuracy of the method is discussed using two test problems. The numerical result shows that the presented method is a successful numerical technique for solving these problems which is showed by presenting the error at different time and space levels.

Keywords: Gardner equation, Harry Dym equation, Quartic B-spline, Collocation method, Von-Neumann stability.

Mathematics Subject Classification: 65L60, 34K20, 49K20.

1 Introduction

The well-known Kortewag-de Veries (KdV) equation (1.1)

\[ u_t + 2\omega u^q u_x + \delta(u^r)_{xxx} = \psi(x) \]  

(1.1)
is an evolutionary nonlinear partial differential equation which is used extensively to model weakly nonlinear long waves, having a certain balance nonlinearity order and dispersion [1]. Recent work have illustrated that although KdV models suites a wide range of parameters of the study, there are cases that the KdV equation is not applicable such as the critical case where the coefficients of the common quadratic nonlinear term in the KdV equation modeling long internal solitary waves vanishes when a symmetrical stratification occurs. It is sufficient to extend the quadratic approximation of the nonlinear term to a high order nonlinear expansion by adding an extra cubic nonlinear term.

This extension gives the combined KdV and modified KdV equation (KdV-MKdV) also known as Gardner equation in the form

\[ u_t + (2\omega u^3 + 3\mu u^2 u_x + \delta (u^2)_{xxx}) = \psi(x, t), \quad \omega, \delta, \mu > 0, \quad (1.2) \]

with the initial and boundary conditions

\[ u(x, 0) = g(x), \quad u(0, t) = g_1(t), \quad u_x(0, t) = g_2(t), \quad u(L, t) = g_3(t), \quad 0 \leq x \leq L, \quad t \geq 0, \quad (1.3) \]

where \( \omega, \mu, \delta, q \) and \( r \) are arbitrary constants. Equation (1.2) models the wave propagation of the bound particle, sound wave and thermal pulse [2]. It also describes a variety of wave phenomena in plasma and solid state [3, 4]. This equation has the same properties as the classical KdV equation, but broadens the range of validity to wider values of the parameters of the internal wave motion [5-13]. In equation (1.2), \( u = u(x, t) \) is the amplitude of the relevant wave mode where \( x \) and \( t \) denote the space variable in the direction of wave propagation and time. The terms \( uu_x \) and \( u^2 u_x \) represents nonlinear wave steepening and the third order derivative term \( u_{xxx} \) represent dispersive wave effects. The coefficients of the nonlinear terms \( \omega \) and \( \delta \) and the dispersive term \( \mu \) are found by the steady oceanic background density and the flow stratification through the linear eignmode of the internal wave. Setting the value of \( \psi(x, t) = u_{xxx}u^2 \) and the value of the constants \( \omega, \mu, \delta = 0 \) gives the known model of Harry Dym equation also known as Dym equation. The equation is is an important dynamical equation which is integrable and finds applications in several physical systems such as representing a system in which dispersion and non-linearity are coupled together. The Harry Dym equation has strong links to the KdV equation and applications of this equation were found to the problems of hydrodynamics [14].

Recently a lot of research work has been presented to find the exact solution of the Gardner equation such as inverse scattering method, Bäcklund transformation, the Wadati trace method, Hirota bilinear forms, Pseudo spectral method, the tanh – sech method [14, 15], the sin – cos method, Riccati expansion method, Adomian decomposition method and Homotopy analysis method [16, 17]. In [18] Krishnan studied the dynamics of solitary waves governed by Gardner’s equation and the fixed point of the solution is obtained. Li and Ma [19] uses the Exp-function method combined with F-expansion method to determine six families of exact solutions of this equation. Also Franca [20] manages to find solutions which are systematically constructed employing the dressing method and deformed vertex operators which takes into account the no vanishing boundary value problem. Likewise, in [21] the authors adapt the \( \left( \frac{g}{\delta} \right) \) Expansion Method to construct the travelling wave solutions involving parameters of Gardner equation. Lastly, a group classification method is used to specify the arbitrary functions of the equation to obtain the maximal symmetry lie algebra to find similarity and invariant solutions [22].

Also, in previous studies of the Gardner equation, the methods used for the derivation of the internal waves solitary wave solutions depends on a prior knowledge of the general mathematical expressions of the expected solutions [23].

In this paper, we aim to adapt quartic B-spline method for solving both Gardner and Harry Dym equations with initial and boundary conditions. The most important feature of this method is that it is easy to implement to linear and nonlinear problems.
2 Application of the collocation method

We introduce the quartic spline space and basis functions to construct an interpolant \( S(x) \) satisfying certain end conditions and then derive several asymptotic expansions to be used in the formulation of the quartic spline collocation method.

Let \( \Delta \equiv \{a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b\} \) be a uniform partition of the interval \([a, b]\) with 8 additional knots outside the region, positioned at:
\[
x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_{0},
\]
\[
x_{N} < x_{N+1} < x_{N+2} < x_{N+3} < x_{N+4},
\]
and \( h = \frac{b-a}{N} \) as a step size. Consider a smooth quartic spline \( S(x) \) that is an element of \( S_4(\Delta) \) \( \equiv \{p(x)/p(x) \in C^3[a, b] \} \). Consider the B-splines basis in \( S_4(\Delta) \) is defined as
\[
B_{m} = \frac{1}{h^4} \begin{cases} 
(x - x_{m-2})^4, & \text{ if } m = 1, \\
(x - x_{m-2})^4 - 5(x - x_{m-1})^4, & \text{ if } 2 \leq m \leq N-3, \\
(x - x_{m-2})^4 - 5(x - x_{m-1})^4 + 10(x - x_{m})^4, & \text{ if } m = N-2, \\
(x_{m+3} - x)^4 - 5(x_{m+2} - x)^4, & \text{ if } m = N, \\
(x_{m+3} - x)^4, & \text{ otherwise}, \\
0, & \text{ otherwise}.
\end{cases}
\]

\[
B_{i-1}(x) = B_{0}(x - (i - 1)h), \quad i = 2, 3, \ldots
\]  

To solve equation (1.2), the \( B_i \) and their first three derivatives, evaluated at the nodal points, are needed. Their coefficients are given in Table 1.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( x_{m-1} )</th>
<th>( x_m )</th>
<th>( x_{m+1} )</th>
<th>( x_{m+2} )</th>
</tr>
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<tbody>
<tr>
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<td>11</td>
<td>11</td>
<td>1</td>
</tr>
<tr>
<td>( hB_m' )</td>
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<td>12</td>
<td>-12</td>
<td>-4</td>
</tr>
<tr>
<td>( h^2B_m'' )</td>
<td>12</td>
<td>-12</td>
<td>-12</td>
<td>12</td>
</tr>
<tr>
<td>( h^3B_m''' )</td>
<td>-24</td>
<td>72</td>
<td>-72</td>
<td>24</td>
</tr>
</tbody>
</table>

An approximate solution \( U(x, t) \) to the exact solution \( u(x, t) \) will be sought in form of an expansion of B-splines:
\[
U(x, t) = \sum_{m=-2}^{N+1} C_m(t)B_m(x),
\]  

where \( C_m(t) \) are unknown time dependent parameters to be determined quartic B-spline collocation form of equations (1.2)-(1.3). The nodal values \( U_m, U_m', U_m'' \) and \( U_m''' \) at the knots \( x_m \) are derived from expression (2.6) and Table 1 in the following form
\[
U_m(x) = c_{m-2} + 11c_{m-1} + 11c_m + c_{m+1},
\]
\[
U_m'(x) = \frac{4}{h}(-c_{m-2} - 3c_{m-1} + 3c_m + c_{m+1}),
\]
\[
U_m''(x) = \frac{12}{h^2}(c_{m-2} - c_{m-1} - c_m + c_{m+1}),
\]
\[
U_m'''(x) = \frac{24}{h^3}(c_{m-2} - 3c_{m-1} + 3c_m - c_{m+1}).
\]  

These knots are used as collocation points. We discretize the time derivative by a difference formula to the space derivative
\[
U_t(x, t) = \frac{1}{\tau}(U^{n+1} - U^n).
\]
Substitute equation (2.11) into equation (1.2), yields
\[
\frac{1}{\tau}(U^{n+1} - U^n) + (2\omega U^q + 3\mu U^{2q})(U_x) + \delta(U_{xxx}) = \psi(x,t),
\]
(2.12)
where \(\tau = \Delta t\) is the time step and the superscripts \(n\) and \(n + 1\) are successive time levels. Hence equation (2.12) can be rearranged as
\[
(U)^{n+1} + \tau (U_x^{n+1})(2\omega(U^q)^{n+1} + 3\mu (U^{2q})^{n+1}) + \tau \delta (U_{xxx})^{n+1} = \tau \psi(x,t) + (U)^n.
\]
(2.13)
To linearize the non-linear term, we use the linearization form given by Rubin and Graves [24]
\[
(U^{2q})^{n+1}(U_x)^{n+1} = (U^{2q})(U_x)^n + 2(U^{2q})(U_x)^n(U^{2q})^{n+1} - 2(U^{2q})^n(U_x)^n
\]
(2.14)
We substitute the non-linear term in equation (2.14) and rearrange the terms and simplifying we get
\[
(U)^{n+1} + 2\omega \tau (U_x)^{n+1}(U^{n+1}) + 3\mu \tau (U^{2q})^n(U_x)^n + 6\mu \tau (U^{2q})^n(U_x)^n(U^{2q})^{n+1} + \tau \delta (U_{xxx})^{n+1} = (U)^n + 6\mu \tau (U^{2q})^n(U_x)^n + \tau \psi(x,t),
\]
(2.15)
then Substituting the approximate solution \(U\) for \(u\) and putting the values of the nodal values \(U\) and its derivatives using equations (2.7)-(2.10) at the knots in equation (2.15) yields the following difference equation with the variables \(C\).
\[
a C_{m+2}^{n+1} + b C_{m+1}^{n+1} + b' c_{m}^{n+1} + a' C_{m-1}^{n+1} = \eta_i, \quad m = 0,1,...,n, \quad i = 0,1,...
\]
(2.16)
Where
\[
\eta_i = U^n + 6\mu \tau (U^{2q})^nU_x^n + \tau \psi(x,t) = d C_{m+2}^n + e C_{m+1}^n + d' C_m^n + e' C_{m-1}^n,
\]
and
\[
a = 1 + W + X + Y + Z, \quad a' = 1 - W - X - Y - Z,
\]
\[
b = 11 + 3(11)^qW + 3X + 11Y + 3Z,
\]
\[
b' = 11 - 3(11)^qW - 3X - 11Y - 3Z,
\]
\[
d = 1 + 3W, \quad d' = 1 - 3W, \quad e = 11 + 9(11)^2qW, \quad e' = 11 - 9(11)^2qW,
\]
\[
W = \frac{-8\omega \tau (1)^q}{h}, \quad X = -\frac{12\mu \tau}{h} (U^{2q})^n, \quad Y = 6\mu \tau (U_x)^n(U^{2q})^n,
\]
\[
Z = \tau \delta (\frac{24}{h^3})^r, \quad N = 3W,
\]
where
\[
(U^{2q})^n = (c_{m+2}^n + 11c_{m+1}^n + 112c_m^n + 2c_{m-1}^n)^2q,
\]
\[
(U_x)^n = \frac{4}{h} (-c_{m+2}^n - 3c_{m+1}^n + 3c_m^n + c_{m-1}^n).
\]
The system (2.16) results in \((N + 1)\) linear equations in \((N + 4)\) unknowns. To solve this system three additional constrains are required. These constrains are obtained from the boundary conditions from equation (1.3). Applying the boundary conditions enables us to add the parameters \(c_{m+2}^{n+1}, c_{m+1}^{n+1}\) and \(c_{m-1}^{n+1}\) in the system, then the system can be reduced to a \((N + 4) \times (N + 4)\) solvable system given by:
\[
AC = D.
\]
(2.17)
Where the matrices \(A, D\) and \(C\) is given by:
The system in equation (2.17) is a system that can be solved by using Thomas algorithm to find the value of the unknown coefficients and then find the approximate solution.

3 Stability analysis

In this section we will investigate the stability analysis by applying Von-Neumann stability analysis. To apply this method, we have linearized the non-linear term $U^2 U_x$ by considering $U^2$ as constant in (2.15).

Now we substitute $C_n h = ε_n \exp(iβh)$ into linearized form of equation (2.16) where $β$ is mode number, $h$ is element size and $= \sqrt{-1}$, we obtain, 

$$e^{n+1} (a e^{i(m+2)φ} + b e^{i(m+1)φ} + b' e^{i(m)φ} + a' e^{i(m-1)φ}) = e^n (d e^{i(m+2)φ} + e e^{i(m+1)φ} + d' e^{i(m)φ} + e' e^{i(m-1)φ}).$$

Here we have taken, $βhφ=q$, dividing both sides by $e^{imφ}$ we get 

$$e^{n+1} (a e^{iφ} + b e^{iφ} + b' + a' e^{-iφ}) = e^n (d e^{iφ} + e e^{iφ} + e' + d' e^{-iφ})$$

and by simplifying, we get, 

$$ε = \frac{A + i B}{C + i D}.$$ 

where $A, B, C$ and $D$ can be put in the form 

$$A = (e + d') \cosφ + d \cos2φ + e' = 2d \cos^2φ + (e + d') \cosφ - d + e' \quad (3.18)$$

$$B = (e - d') \sinφ + d \sin2φ = (2d \cosφ + e - d') \sinφ \quad (3.19)$$

$$C = (b + a') \cosφ + a \cos2φ + b' = 2a \cos^2φ + (b + a') \cosφ - a + b' \quad (3.20)$$

$$D = (b - a') \sinφ + a \sin2φ = (2a \cosφ + b - a') \sinφ \quad (3.21)$$

Then after substitution and simplification we find that 

$$|ε| = A^2 + B^2 - C^2 - D^2 \leq 1$$

which implies that the scheme is unconditionally stable.

4 Test problems

The numerical method described in the previous section is tested on a test problem for getting solution of the Gardner equation in order to demonstrate the robustness and numerical accuracy [25].

We use the $L_∞$ error norm to measure error between the analytical and numerical solutions which is calculated from the following formula 

$$L_∞ = |U - U_N|_∞ = \max_j |U_j - (U_N)_j|.$$
Where $U_j$ is the exact solution at any level $j$ and $(U_N)_j^n$ is the approximate solution at the same level $j$.

4.1. Test problem (1) Gardner Equation
Considering Gardner equation given by equation (1.2) by assigning the values of the coefficients as $p = q = r = 1$, $\mu = \omega = \delta = 1$ and $\phi(x) = 0$ which gives the equation in the form

$$u_t + 2uu_x + u_{xxx} = 0,$$

with the following initial and boundary conditions as

Initial condition:
$$u(x, 0) = \frac{E}{F + \cosh(Gx)},$$

Boundary conditions:
$$u(0, t) = \frac{E}{F + \cosh(-v t)}, \quad u_x(0, t) = \frac{GE \sinh(Gt v)}{(F + \cosh(Gv t))^2}, \quad u(1, t) = \frac{E}{F + \cosh(G(x - v))}.$$

And the constants are as follows

$$E = \frac{6v}{\sqrt{4\omega^2 + 18\delta v}}, \quad F = \sqrt{v}, \quad G = \frac{2\omega}{\sqrt{4\omega^2 + 18\delta v}}, \quad v = \frac{1}{2}.$$

This problem has the following exact solution

$$u(x, t) = \frac{E}{F + \cosh(G(x - v t))}.$$

Tables 2, 3 and 4 shows the absolute error between the approximate and exact solution at different time and space level. The numerical results prove the advantage of this method. Also, can be seen from the numerical results that the ability of the method of providing a high accurate numerical solution of any system. We notice from the figures that changing the time interval does not affect the pattern of the solution. Also, we notice that the approximate and the exact solutions are identical.

<table>
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<th>0.2</th>
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<th>0.6</th>
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<td>0.00E+00</td>
<td>0.00E+00</td>
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<tr>
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</tr>
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Table 3: Absolute maximum error for $0 < x < 1$, $0 < t < 10$

<table>
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</table>

Figure 1: Problem 1 (a) Approximate solution, (b) Exact solution, (c) $L_{\infty}$ error for $0 < x < 1$, $0 < t < 10$. 
4.1. Test Problem (2) Harry Dym equation

Considering Harry Dym equation given by setting the values of the coefficients as $\mu = \omega = \delta = 0$ and $\psi(x) = u_{xxx}u^3$ in equation (1.2) which gives the equation in the form $u_t - u^3 u_{xxx} = 0$,

with the following initial and boundary conditions

$$u(x, 0) = \left( a - \frac{3\sqrt{b}}{2} x \right)^{\frac{2}{3}} , \quad u(0, t) = \left( a - \frac{3\sqrt{b}}{2} c t \right)^{\frac{2}{3}}$$

$$u_x(0, t) = \frac{-1}{\sqrt{b} \left( a - \frac{3\sqrt{b}}{2} c t \right)^{\frac{1}{3}}} , \quad u(1, t) = \left( a - \frac{3\sqrt{b}}{2} (1 + c t) \right)^{\frac{2}{3}}.$$ 

Following the exact solution as from [27]

$$u(x, t) = u(1, t) = \left( a - \frac{3\sqrt{b}}{2} (x + c t) \right)^{\frac{2}{3}}$$

Tables 4 and Table 5 gives the error between the approximate and exact solution of test problem 2.
Table 4: Absolute maximum error for $0 < x < 1$, $0 < t < 0.1$.

<table>
<thead>
<tr>
<th>X/T</th>
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<th>0.04</th>
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<tr>
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</table>

Figure 3: Problem 2 (a) Approximate solution, (b) Exact solution, (c) $L_\infty$ error for $0 < x < 1$, $0 < t < 0.1$. 
Table 5: Absolute maximum error for $0 < x < 1$, $0 < t < 1$.

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</tbody>
</table>

Figure 4: Problem 2 (a) Approximate solution, (b) Exact solution, (c) $L_\infty$ error for $0 < x < 1$, $0 < t < 1$. 
5 Conclusion

In this paper, an effective numerical treatment for the nonlinear Gardner and Harry Dym equations is proposed using a collocation method using usual finite scheme for the time discretization and the quartic B-spline functions for space discretization. The application of the quartic B-spline method presented in this paper is simple and straightforward. The algorithm described above works for a large class of linear and nonlinear problems. The solution obtained is presented at various time intervals shows the same characteristics as given in the literature. The stability analysis of the method is shown to be unconditionally stable. The obtained numerical solutions maintain a good agreement with the exact solution.

References

https://doi.org/10.1002/mma.1670150202

https://doi.org/10.1016/j.jsc.2003.09.004

https://doi.org/10.1023/A:1026615919186


https://doi.org/10.1016/0167-2789(94)90299-2

https://doi.org/10.1111/1467-9590.00098

https://doi.org/10.5194/npg-9-221-2002

https://doi.org/10.1016/S0165-2125(02)00093-8
https://doi.org/10.1017/MAF05016

https://doi.org/10.1175/1520-0485(1997)027<0871:AMOIT>2.0.CO;2

https://doi.org/10.1029/1999JC900144

https://doi.org/10.1007/0-306-48024-7_2

https://doi.org/10.1103/PhysRevA.44.6490

https://doi.org/10.1088/0031-8949/54/6/003

https://doi.org/10.1088/0031-8949/54/6/004

https://doi.org/10.1016/j.amc.2004.09.033


https://doi.org/10.1007/s11071-010-9928-7


https://doi.org/10.1088/1751-8113/45/1/015207


