Another Method for Defuzzification Based on Signal/Noise Ratios and its Applications in Comparing DMUs

R. Saneifard* 

Department of Mathematics, Urmia Branch, Islamic Azad University, Urmia, Iran.

Abstract

A new method for the defuzzification of fuzzy numbers is developed in this paper. It is well-known, defuzzification methods allow us to find aggregative crisp numbers or crisp set for fuzzy numbers. But different fuzzy numbers are often converted into one crisp number. In this case the loss of essential information is possible. It may result in inadequate final conclusions, for example, expert estimation problems, prediction problems, etc. Accordingly, the necessity to develop a method for the defuzzification of fuzzy numbers, allowing us to save their informative properties has arisen. The purpose of this paper is to develop such a method. In this paper, we present a new approach for defuzzification using the $\gamma$-cut, the belief features and the signal/noise ratios of fuzzy numbers, where $\gamma \in [0,1]$. The proposed method can overcome the drawbacks of other methods. We start with the definition of the signal/noise ratios for fuzzy numbers. The the signal/noise ratios for fuzzy number is defined as the set of weighted points of all unimodal numbers, that belong to this number. Some propositions and examples about the signal/noise ratios properties are offered.

Keywords: Ranking; Fuzzy number; L-R type; Defuzzification; Signal/noise ratios.

1 Introduction

As it is well-known, defuzzification methods convert a fuzzy number into a crisp real number. But often different fuzzy numbers are converted into one crisp number. For example, according to the definition of weighted fuzzy arithmetic, two normalized symmetrical triangular numbers with different fuzzy widths are converted into one crisp number[4]. This may not present a problem to solve a number of practical tasks, however, for example, in decision making problems and some other problems the necessity arises to find aggregative indexes that will possibly accumulate different bounds of input fuzzy numbers. Moreover, while making regression models, it is easier to operate with aggregative indexes than with the proper fuzzy numbers. The purpose of this paper is to develop a new method for the defuzzification of fuzzy numbers, that will allow to keep their informative properties. This paper starts with the definitions of the signal/noise ratios for fuzzy numbers. Then, three propositions about the the signal/noise ratios properties are proved.

*Corresponding author. Email address: srsaneifard@yahoo.com
2 Basic Definitions and Notations

In this paper, we assume that the reader is familiar with basics of fuzzy set theory and fuzzy logic in the broad sense [1, 2, 5, 3].

Definition 2.1. Let $X$ be a universe set. A fuzzy set $A$ of $X$ is defined by a membership function $\mu_A(x) \to [0, 1]$, where $\mu_A(x)$, $\forall x \in X$, indicates the degree of $x$ in $A$.

Definition 2.2. A fuzzy subset $A$ of universe set $X$ is normal iff $\sup_{x \in X} \mu_A(x) = 1$, where $X$ is the universe set.

Definition 2.3. A fuzzy set $A$ is a fuzzy number iff $A$ is normal and convex on $X$.

Definition 2.4. For fuzzy set $A$ Support function is defined as follows:

$$\text{supp}(A) = \{x | \mu_A(x) > 0\},$$

where $\{x | \mu_A(x) > 0\}$ is the closure of set $\{x | \mu_A(x) > 0\}$.

Definition 2.5. A L-R fuzzy number $A = (m, n, \sigma, \beta)_{LR}$, $m \leq n$, is defined as follows:

$$\mu_A(x) = \begin{cases} L(\frac{x - m}{\sigma}), & -\infty < x < m, \\ 0, & m \leq x \leq n, \\ R(\frac{x - n}{\beta}), & n < x < +\infty. \end{cases}$$

Where $\sigma$ and $\beta$ are the left-hand and right-hand spreads. In the closed interval $[m, n]$, the membership function is equal to 1. $L(\frac{x - m}{\sigma})$ and $R(\frac{x - n}{\beta})$ are non-increasing functions with $L(0) = 1$ and $R(0) = 1$, respectively. Usually, for convenience, they are, respectively, denoted as $\mu_{LA}(x)$ and $\mu_{RA}(x)$. It needs to point out that when $L(\frac{x - m}{\sigma})$ and $R(\frac{x - n}{\beta})$ are linear functions and $m < n$, fuzzy number $A$ denotes trapezoidal fuzzy number. When $L(\frac{m - x}{\sigma})$ and $R(\frac{n - x}{\beta})$ are linear functions and $m = n$, fuzzy number $A$ denotes unimodal fuzzy number.

This definition is very general and allows the quantification of quite different types of information; for instance, if $A$ is supposed to be a real crisp number for $m \in \mathbb{R}$,

$$A = (m, m, 0, 0)_{LR}, \forall L, \forall R$$

If $A$ is a crisp interval,

$$A = (a, b, 0, 0)_{LR}, \forall L, \forall R$$

and if $A$ is a trapezoidal fuzzy number, $L(x) = R(x) = \max(0, 1 - x)$ is implied.

Let $F$ denote the space of L-R fuzzy numbers, therefore, in this article it is assumed that, the fuzzy number $A \in F$ is presented by means of the following representation:

$$A = \bigcup_{\alpha \in [0, 1]} (\alpha, A_\alpha)$$

(2.1)

where

$$\forall \alpha \in [0, 1] : A_\alpha = [L_\alpha(\alpha), R_\alpha(\alpha)] \subset (-\infty, \infty)$$

(2.2)

Here, $L : [0, 1] \to (-\infty, \infty)$ is a monotonically non-decreasing and $R : [0, 1] \to (-\infty, \infty)$ is a monotonically non-increasing left-continuous functions. The functions $L(\cdot)$ and $R(\cdot)$ express the left and right sides of a fuzzy number, respectively [5, 6]. In other words,

$$L(\alpha) = \mu^{-1}_L(\alpha), \quad R(\alpha) = \mu^{-1}_R(\alpha),$$

(2.3)

where $L(\alpha) = \mu^{-1}_L(\alpha)$, and $R(\alpha) = \mu^{-1}_R(\alpha)$, denote quasi-inverse functions of the increasing and decreasing parts of the membership functions $\mu(t)$, respectively. As a result, the decomposition representation of the fuzzy number $A$, called the L-R representation, has the following form:

$$A = \bigcup_{\alpha \in [0, 1]} (\alpha, [L_\alpha(\alpha), R_\alpha(\alpha)]).$$
Definition 2.6. [7] A function $f : [0, 1] \rightarrow [0, 1]$ symmetric around $\frac{1}{2}$, i.e. $f(\frac{1}{2} - \alpha) = f(\frac{1}{2} + \alpha)$ for all $\alpha \in [0, \frac{1}{2}]$, which reaches its minimum in $\frac{1}{2}$, is called the bi-symmetrical weighted function. Moreover, the bi-symmetrical weighted function is called regular if

(1) $f(\frac{1}{2}) = 0,$
(2) $f(0) = f(1) = 1,$
(3) $\int_{0}^{1} f(\alpha) d\alpha = \frac{1}{2}.$

One can, of course, propose many regular bi-symmetrical weighted functions and hence obtain different bi-symmetrical weighted distances. Further on we consider mainly the following function

$$f(\alpha) = \begin{cases} 1 - 2\alpha & \text{when } \alpha \in [0, \frac{1}{2}], \\ 2\alpha - 1 & \text{when } \alpha \in [\frac{1}{2}, 1]. \end{cases}$$

Definition 2.7. [8, 9] Let $A = (m, m, \sigma, \beta)_{LR}$ be a unimodal $L$-$R$ fuzzy number and $A_{\alpha} = [L_{A}(\alpha), R_{A}(\alpha)]$ be its $\alpha$-cut sets. Assume that a decision maker wants to determine the ranking order of $m$ fuzzy numbers $A_{1}, A_{2}, \cdots, A_{m}$. The $k$th $\gamma$-cut $A_{i}^{k}$ of fuzzy number $A_{i}$ is defined as follows:

$$A_{i}^{k} = \{ x | f_{n}(x) \geq \gamma, x \in X \},$$

where $n$ denotes the number of $\gamma$-cuts.

The minimal value $l_{i,k}$ and the maximal value $r_{i,k}$ of the $k$th $\gamma$-cut of the fuzzy number $A_{i}$ are defined as follows:

$$l_{i,k} = \inf_{x \in X} \{ x | f_{n}(x) \geq \gamma \}.$$  
$$r_{i,k} = \sup_{x \in X} \{ x | f_{n}(x) \geq \gamma \}.$$  

respectively. The maximal barrier $U$ and the minimal barrier $L$ of the $m$ fuzzy numbers $A_{1}, A_{2}, \cdots, A_{m}$ are defined as follows:

$$U = \max_{i \in \{1, 2, \cdots, m\}} \{ x | x \in A_{i}^{\gamma}, 0 \leq \gamma \leq h_{A_{i}}, 1 = 1, 2, \cdots, m \},$$

$$L = \min_{i \in \{1, 2, \cdots, m\}} \{ x | x \in A_{i}^{\gamma}, 0 \leq \gamma \leq h_{A_{i}}, 1 = 1, 2, \cdots, m \}.$$  

where $A_{i}^{\gamma}$ denotes the $\gamma$-cut of the fuzzy number $A_{i}$ and $h_{A_{i}}$ denotes the height of $A_{i}$ defined as follows:

$$h_{A_{i}} = \sup_{x \in X} f_{A_{i}}(x).$$

The signal/noise ratio $\eta_{i,k}$ of the $k$th $\gamma$-cut of the fuzzy number $A_{i}$ used in the proposed method is defined as follows:

$$\eta_{i,k} = \frac{m_{i,k} - L}{\delta_{i,k} + c},$$

where

$$m_{i,k} = \frac{r_{i,k} - l_{i,k}}{2},$$

$$\delta_{i,k} = \frac{r_{i,k} + l_{i,k}}{2},$$

and $c$ is a constant.
where \( m_{i,k} \) and \( d_{i,k} \) denote the middle-point and the spread of \( A_i^k \), respectively, defined as follows:

\[
m_{i,k} = \frac{r_{i,k} + l_{i,k}}{2},
\]

\[
d_{i,k} = r_{i,k} - l_{i,k}.
\]

\( L \) denotes the minimal barrier of the \( m \) fuzzy numbers \( A_1, A_2, \ldots, A_m \) defined by Eq. (2.9), \( c \) is a parameter, and \( c > 0 \). The parameter \( c > 0 \) is used to avoid the case that if the fuzzy number \( A_i \) is the crisp value "0", the signal/noise ratio will be indeterminate. From Eq. (2.11), we can find that the larger the value of \( c \), the smaller the influence of \( d_{i,k} \) on the signal/noise ratio \( \eta_{i,k} \). Therefore, we think that the influence of \( d_{i,k} \) on \( \eta_{i,k} \) should be smaller than the influence of \( m_{i,k} \) on \( \eta_{i,k} \). The value of \( c \) should be greater than the value of \( R - L \) in order to avoid the special case that if we want to obtain the ranking order of two equal crisp values \( A_1 \) and \( A_2 \), the values of \( R - L \) and \( d_{i,k} \) of the \( k \)th \( \gamma \)-cut of the fuzzy number \( A_1 \) and \( A_2 \) will be all zero and the signal/noise ratio will be indeterminate or undefined, where \( \gamma_k \in [0,1] \). In the following, we present a new approach for comparing fuzzy numbers based on the distance method. The method not only considers the signal/noise ratio of a fuzzy number, but also considers the minimum crisp value of fuzzy numbers. The proposed method for ranking fuzzy numbers \( A_1, A_2, \ldots, A_m \) is now presented as follows:

Use the point \((SN(A_j), 0)\) to calculate the ranking value \( sn/r(A_j) = E(SN(A_j), x_{max}) \) of the fuzzy numbers \( A_j \), where \( A_j \), for \( 1 \leq j \leq m \), as follows:

\[
E?(SN(A_j), x_{max}) = \|SN(A_j) - x_{max}\|
\]

From formula (2.14), we can see that \( sn/r(A_j) = E(SN(A_j), x_{max}) \) can be considered as the Euclidean distance between the point \((SN(A_j), 0)\) and the point \((x_{max}, 0)\). We can see that the larger the value of \( sn/r(A_j) \), the better the ranking of \( A_j \), where \( 1 \leq j \leq m \). When ranking \( n \) fuzzy numbers \( A_1, A_2, \ldots, A_m \), the minimum crisp value \( x_{\text{min}} \) is defined as:

\[
x_{\text{max}} = \max\{x \in \text{Domain}(A_1, A_2, \ldots, A_m)\}.
\]

The index \( SN(A_j) \) of fuzzy numbers \( A_j \) is calculated as \( SN(A_j) = \frac{h_{\gamma_k} \sum_{i=1}^{n} h_i \times h_{\gamma_k}}{\sum_{i=1}^{n} h_i} \), where \( \gamma = h_{\alpha_k} \times \frac{\gamma}{\alpha} \), \( k \in \{1, 2, \ldots, n\} \), \( n \in N \), and \( n \) denotes the number of \( \gamma \)-cuts.

**Example 2.1.** Consider two triangular fuzzy numbers \( A = (2, 2, 2)_{LR} \) and \( B = (2, 2, 1, 1) \). The signal/noise ratios for fuzzy numbers \( A \) and \( B \), we shall designated accordingly as \( SN(A) \) and \( SN(B) \). According to [2], we have \( B \prec A \).

**Example 2.2.** Let \( A = (0.1, 0.3, 0.3, 0.5; 1) \) and \( B = (0.2, 0.3, 0.3, 0.4; 1) \) be two generalized trapezoidal fuzzy numbers. Find \( a = \min(0.1, 0.2) = 0.1 \).

For \( \alpha = 0 \), \( E(SN(A), x_{\text{max}}) = 0.4 \), \( E(SN(B), x_{\text{max}}) = 0.35 \), \( RM_a(A) = 0.4 - 0.1 = 0.3 \) and \( RM_a(B) = 0.35 - 0.1 = 0.25 \). So \( A \succ B \).

For \( \alpha = 1 \), \( E(SN(A), x_{\text{max}}) = 0.2 \), \( E(SN(B), x_{\text{max}}) = 0.25 \), \( RM_a(A) = 0.2 - 0.1 = 0.1 \) and \( RM_a(B) = 0.25 - 0.1 = 0.15 \). So \( A \prec B \).

For \( \alpha = \frac{1}{2} \), \( E(SN(A), x_{\text{max}}) = 0.3 \), \( E(SN(B), x_{\text{max}}) = 0.3 \), \( RM_a(A) = 0.3 - 0.1 = 0.2 \) and \( RM_a(B) = 0.3 - 0.1 = 0.2 \). So \( A \sim B \).

**Example 2.3.** Consider the three triangular fuzzy numbers \( A = (6, 1, 1) \), \( B = (6, 0.1, 1) \) and \( C = (6, 0, 1) \). By using our method, \( E(SN(A), x_{\text{max}}) = 1 + \frac{\alpha}{2} \), \( E(SN(B), x_{\text{max}}) = \frac{\alpha}{2} + 0.55 \), \( E(SN(C), x_{\text{max}}) = \frac{\alpha}{2} + 0.5 \) and with \( a = \min(6, 6, 6) = 6 \), we have \( RM_a(A) = \frac{\alpha}{2} - 5 \), \( RM_a(B) = \frac{\alpha}{2} - 5.45 \), \( RM_a(C) = \frac{\alpha}{2} - 5.5 \). So \( A \succ B \prec C \).

**Example 2.4.** Let \( A = (0.2, 0.4, 0.6, 0.8; 0.35) \) and \( B = (0.1, 0.2, 0.3, 0.4; 0.7) \) be two generalized trapezoidal fuzzy numbers.

For \( a = \min(0.2, 0.1) = 0.1 \), and \( \alpha = 0 \), then \( E(SN(A), x_{\text{max}}) = 0.245 \), \( E(SN(B), x_{\text{max}}) = 0.1225 \), \( RM_a(A) = 0.245 - \)
0.1 = 0.145 and $RM_a(B) = 0.1225 - 0.1 = 0.0225$. So $A \succ B$.

Also, for $\alpha = 1$ we have, $RM(A) = 0.105$, $RM(B) = 0.0525$, $RM_a(A) = 0.105 - 0.1 = 0.005$ and $RM_a(B) = 0.0525 - 0.1 = -0.0475$. So $A \succ B$.

For $\alpha = \frac{1}{2}$, $RM(A) = 0.175$, $RM(B) = 0.0875$, $RM_a(A) = 0.175 - 0.1 = 0.075$ and $RM_a(B) = 0.0875 - 0.1 = -0.0125$. So $A \succ B$.

It can be observed that signal/noise ratios (aggregative crisp numbers) are the same for two triangular fuzzy numbers with different fuzzy widths, while the signal/noise ratios for these fuzzy numbers are different. The greater the fuzzy widths, the greater the signal/noise ratios.

3 Conclusion

The fuzzy number defuzzification method with regular weighted intervals was developed in this article. The developed method was suggested to be used in situations where it is necessary to accumulate more information about fuzzy numbers than aggregative point crisp indexes contain, when there is no need to get only aggregative numbers. The numerical example demonstrated that the developed method can be used for the successful accumulation different bounds of fuzzy numbers.

Acknowledgment

This paper is based on the Masters project at the Department of Mathematics, Urmia Branch, Islamic Azad University, Urmia in Iran, under graduate research fellowship and fundamental research grant scheme.

References


