Solving Duffing equation using an improved semi-analytical method

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Abstract
Since achieving the precise analytical solution of most nonlinear problems is not possible, numerical methods are meant to be used. In fact a very limited number of nonlinear problems have the exact analytical solution. Let us consider some other methods which are known as semi-analytical methods and applied to solve nonlinear problems. In these methods by using a primary approximation and an iterative process, the approximate solution can be provided, so that the iterations converge to the desired solution. Methods which have been employed and support researchers in this field are categorized as "Adomian decomposition method"(ADM), "Homotopy perturbation method"(HPM), "Homotopy analysis method"(HAM), and "He variational iteration method"(VIM). The aim of this paper is to suggest an improvement of reliability of Adomian decomposition method to obtain a solution to Duffing equation. In details, Adomian decomposition method using modified Legendre polynomials has been applied in the present work that allows researchers to provide approximation of the solution with high accuracy. This method is tested for example. Moreover the obtained numerical results will be presented.

Keywords: Adomian decomposition method, modified Legendre polynomials, the equation of Duffing.

1 Introduction
In last years, for solving ordinary and partial differential equations, linear and nonlinear integral equations, Adomian decomposition method has been an interesting subject for researchers to work on. [1, 2]. The proposed algorithm of this method has been tested on various equations and aims to find the successive terms of finite series solution and get an unknown amount which assumes that each term of the series can easily be calculated. In particular, the beauty of the Adomian decomposition method is its accuracy and speediness in finding numerical solution of a convergent series. The suggested method is relatively free from errors. Subsequently, the enhanced form of the method can be efficiently employed for solving linear and nonlinear ordinary differential equations.

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2 Preliminaries and notations

Definition 2.1. Modified Legendre polynomials are a set of functions similar to Legendre polynomials, but they are defined on interval (0, 1). For example some of the first functions of modified Legendre polynomials are given by:

\[ P_n(x) \]


\[ n = 0, 1 \]

\[ n = 1, 2x - 1 \]

\[ n = 2, 6x^2 - 6x + 1 \]

\[ n = 3, 20x^3 - 30x^2 + 12x - 1. \]

(2.1)

3 Main section

In the 1969, George Adomian introduced a mighty method. This method is known as the standard Adomian decomposition method (ADM). In recent years many studies have been implemented in this matter [3, 4]. This technique is based on decomposing the unknown function to the series of the infinite components which is stated as \( u(x) = \sum_{n=0}^{\infty} u_n(x) \), the components \( u_n(x) \) are determined by a recursive relation.

The crucial aspect of the method is determining the series \( u_0(x) \), \( u_1(x) \), \( u_2(x) \), … which gives the solution to the original problem. These unknown series are simply defined by a recursive relation. To do this, we substitute the unknown function of ordinary differential equation (or partial) for the above series. Therefore, the general differential equation can be written as following:

\[ Lu + Ru + Nu = g(x), \ldots \]

(3.2)

Where \( L \) is the highest order of derivative which is assumed reversible, \( R \) is the operator of linear derivative with the lower order than \( L \), \( N \) is a nonlinear operator and the term \( g(X) \) is the source term. For simplicity, we shall consider \( L^{-1} \) as the inverse operator to \( L \). Applying \( L^{-1} \) to both sides of equation (2.1) we find that:

\[ u = L^{-1}(g(x)) - L^{-1}(Ru) - L^{-1}(Nu). \]

(3.3)

For brevity, we assume \( f = L^{-1}(g(x)) \) therefore

\[ u = f - L^{-1}(Ru) - L^{-1}(Nu), \]

(3.4)

Providing the default conditions \( \varphi(X) \) is available. According to standard Adomian decomposition method, the solution of \( u(x) \) is defined as a series \( u = \sum_{i=0}^{\infty} u_i \), and for nonlinear term, this method offers the following series:

\[ Nu = \sum_{i=0}^{\infty} A_i, \]

(3.5)

where each \( A_i \) is an Adomian polynomial. Considering equation (3.3), we find that:

\[
\begin{align*}
  u_0 &= L^{-1}(g) + \varphi(x), \\
  u_1 &= -L^{-1}(Ru_0) - L^{-1}(Nu_0), \\
  u_2 &= -L^{-1}(Ru_1) - L^{-1}(Nu_1), \\
  \vdots
\end{align*}
\]
Now, given the above polynomials, the following general form is calculated:
\[ u_{i+1} = -L^{-1}(Ru_i) - L^{-1}(Nu_i), \quad i \geq 0. \]  

(3.6)

Subject to \( i \geq 0 \) Having (3.6), \( u(x) \) is explicitly or implicitly decomposed into an infinite series. As \( n \to \infty \), calculating the \( n \)-th approximation \( \psi_n = \sum_{k=0}^{n-1} u_k \) tends to \( u = \sum_{k=0}^{\infty} u_k \), that is, \( \lim_{n \to \infty} \psi_n = u \). Also we need a term for \( A_n \). Thus we get:

\[ A_0 = F(u_0), \]
\[ A_1 = u_1 F'(u_0), \]
\[ A_2 = u_2 F'(u_0) + \left( \frac{1}{2!} \right) u_1^2 F''(u_0), \]
\[ A_3 = u_3 F'(u_0) + u_1 u_2 F'(u_0) + \left( \frac{1}{3!} \right) u_1^3 F''(u_0), \]
\[ A_4 = u_4 F'(u_0) + \left[ \left( \frac{1}{2!} \right) u_2^2 + u_1 u_3 \right] F''(u_0) + \left( \frac{1}{4!} \right) u_1^4 F^{(iv)}(u_0). \]

(3.7)

Continuing in this manner, Polynomials can be extended to other similar process, Subject to \( F(u) = Nu \)

Generally, to implementing the Adomian decomposition method, source term is expended to the Taylor series as given by:

\[ g(x) \approx \sum_{i=0}^{\infty} g_i(x). \]  

(3.8)

Where \( i \) is an arbitrary natural number. In this paper, we propose that (3.8) expand to Legendre series as given by:

\[ g(x) \approx \sum_{i=0}^{\infty} C_i P_i(x). \]  

(3.9)

Where \( P_i(x) \) is general mode of Legendre polynomial of second order which can be written as:

\[ P_i(x) = \frac{2i-1}{i} \cdot x \cdot P_{i-1}(x) - \frac{i-1}{i} \cdot P_{i-2}(x), \quad i \geq 1. \]  

(3.10)

Thus we have:

\[
\begin{cases}
  u_0 = L^{-1}(C_0 P_0(x) + C_1 P_1(x) + \ldots + C_i P_i(x)) + \varphi(x), \\
  u_i = -L^{-1}(Ru_0) - L^{-1}(Nu_i), \\
  u_{i+1} = -L^{-1}(Ru_i) - L^{-1}(Nu_{i+1}), \\
  \vdots 
\end{cases}
\]  

(3.11)

4 Numerical examples

Consider the following Duffing equation

Where \( 0 \leq x \leq 1 \)

\[ u'' + 3u - 2u^3 = \cos(x) \cdot \sin(2x), \]  

(4.12)

Subject to \( u(0) = 0, \quad u'(0) = 1. \)
In this case, the Analytical solution is given by [2]:

\[ u(x) = \sin(x) \]  

(4.13)

According to equation (3.3) we get:

\[ u = L^{-1}(\cos(x) \cdot \sin(2x)) - L^{-1}(3u) + L^{-1}(2u^3), \]

Where \( L^{-1} = \int_{0}^{1} \int_{0}^{1} (\cdot) \). Now if we consider extended Legendre function \( g(x) \) respect to \( \lambda = n = 7 \), in this case, we have:

\[ g(x) \approx \sum_{i=0}^{7} C_i P_i(2x - 1), \quad 0 \leq x \leq 1, \]

(4.14)

Where:

\[ C_0 = \frac{\int_{-1}^{1} g(0.5x + 0.5) \cdot P_0(x)dx}{\int_{-1}^{1} P_0^2(x)dx}, \]

(4.15)

\[ C_i = \frac{\int_{-1}^{1} g(0.5x + 0.5) \cdot P_i(x)dx}{\int_{-1}^{1} P_i^2(x)dx}, \quad i = 1, 2, \ldots, 7. \]

So, the equation (4.14) is given by:

\[ g(x) \approx -0.00000549791076 + 2.00039948398x + \cdots -0.0157014x^2. \]

(4.16)

And, the following results are obtained as:

\[ u_0 = L^{-1}(0.00000549791076 + 2.00039948398x + \cdots -0.0157014x^2) + u(0) + u'(0)x \]

\[ = x - 0.00000274895538x^2 + 0.333399913997x^3 + \cdots -0.000218075x^9, \]

\[ u_1 = -0.5x^3 + 6.87238845 \times 10^{-7}x^4 + 0.0499900129004x^5 + \cdots, \]

\[ u_2 = 0.075x^4 - 6.87238845 \times 10^{-9}x^6 + \cdots, \]

\[ u_3 = -0.0053571428571x^7 \ldots \]

\[ \vdots \]

\[ u_7(x) = \sum_{i=0}^{6} u_i = x - 0.00000274895538x^2 - 0.166600086003x^3 - \cdots -0.000583917325605x^7 + \cdots \]

The profile of the absolute error, the approximate solution and exact solution of \( u_L(x) \) have been determined and recorded in Table 1.

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5 Conclusion

In our work, Proposed improvement to Adomian decomposition method according to the numerical results showed that nonlinear terms of given equations here can be employed in this method. In this paper, the obtained Numerical results are very encouraging and reveal that the modified method is very accurate, simple, effective and powerful. And it is also an efficient technique for solving different kinds of equations specially those which given here. A reliable modification was proposed, and the modified method was employed to solve the nonlinear equations; In conclusion, of all those mentioned method Adomian method yields an exact analytical solution. Proposed improvement to Adomian decomposition method according to the numerical results showed that nonlinear terms of given equations here can be employed in this method. This study showed also, the higher speed of the convergent of Adomian decomposition method for the solution. The method can be also extended to other nonlinear problems. This method also requires less time to calculate the series. however, it gives an accurate approximation.

References


