A solution of dual fully fuzzy linear system of equations

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1 Introduction

Using the embedding approach Friedman et. al. in [10] proposed a general model to solving such a fuzzy linear systems $A\tilde{x} = \tilde{b}$, where $A$ is a crisp matrix and $\tilde{b}$ is a fuzzy number vector. They have used the parametric form of fuzzy numbers and replace the original fuzzy linear system by $2n \times 2n$ crisp system. Asady et al. [5], who merely discussed the full row rank system, used the same method to solve the $m \times n$ fuzzy linear system for $m \leq n$. Zheng and wang [15, 16] discussed the solution of the general $m \times n$ consistent and inconsistent fuzzy linear system.

On the fuzzy linear system of dual equations $A\tilde{x} = B\tilde{x} \oplus \tilde{c}$, where $A$ and $B$ are real square matrices, Friedman et. al. [11] investigated the existence of the solutions. They claimed that this type of system cannot be replaced by a fuzzy linear system $(A - B)\tilde{x} = \tilde{b}$.

The system of linear equations $\tilde{a}_{ij}\tilde{x}_j = \tilde{b}_i$ where $\tilde{a}_{ij}$ and $\tilde{b}_i$ are the elements of the matrix $A$ and vector $b$ respectively are fuzzy numbers, is called a fully fuzzy linear systems (FFLS). FFLS has been studied by several authors. Moloudzadeh et. al. in [1, 12, 13] have studied some methods for solving FFLS. They in [1, 13] have represented fuzzy numbers in LR form and applied approximate operations between fuzzy numbers to find arbitrary solutions of FFLS, so finding the solutions of FFLS is transformed into finding the solutions of two crisp systems. So many works have been done to find the solution of the FFLS (see [3, 4, 6, 7, 8]). Muzzioli et. al. in [14] studied FFLS of the form $A_1x + b_1 = A_2x + b_2$. The purpose of this paper is to present a solution of an arbitrary dual fully fuzzy linear systems (DFFLS). To do this, the original $n \times n$ dual fully fuzzy linear system $A \otimes \tilde{x} = B \otimes \tilde{x} \oplus \tilde{c}$ is transformed into two crisp linear systems, then obtained the solution of this two systems.

This paper is organized as follows: In section 2, the basic results of the fuzzy numbers and fuzzy calculus are discussed and a summary of the fuzzy matrix will be illustrated. In section 3, an arbitrary DFFLS are discussed and
our method for solving DFFLS, \( \tilde{A}x = \tilde{B}c \oplus \tilde{c} \) is introduced. The presented method are shown by some examples in section 4 and conclusion are drawn in section 5.

2 Basic Definitions and Notations

In this section, we give some necessary definitions and notations which will be used throughout the paper.

Definition 2.1. Let \( X \) be a nonempty set. A fuzzy set \( u \) in \( X \) is characterized by its membership function \( u : X \rightarrow [0, 1] \). Thus \( u(x) \) is interpreted as the degree of membership of an element \( x \) in the fuzzy set \( u \) for each \( x \in X \).

Let denote by \( E \) the class of fuzzy subsets of the real axis (i.e. \( u \in \mathbb{R} \)). Among the infinite number of possible fuzzy sets in \( u \in X \) that qualify as fuzzy numbers, some types of membership functions \( u(x) \) are of particular importance, especially with respect to the use of fuzzy numbers in applied fuzzy arithmetic. In this paper we will utilize LR fuzzy number.

Definition 2.2. A fuzzy number \( \tilde{A} \) is called a LR fuzzy number if its membership function \( \mu_{\tilde{A}} : R \rightarrow [0, 1] \) has the following form:

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
L(\frac{x-a}{\alpha}) & , x \leq a, \alpha > 0, \\
R(\frac{x-b}{\beta}) & , x \geq a, \beta > 0.
\end{cases}
\]  

(2.1)

where \( a \) is the mean value of \( \tilde{A} \), \( L : [0, 1] \rightarrow [0, 1] \) and \( R : [0, 1] \rightarrow [0, 1] \) are continuous and non-increasing shape functions with \( L(0) = R(0) = 1 \) and \( L(1) = R(1) = 0 \). The LR fuzzy number \( \tilde{A} \) as described above will be represented as \( \tilde{A} = (a, \alpha, \beta)_{LR} \). Here \( L \) and \( R \) are called as the left and right reference functions, \( a \) is called a mean value, \( \alpha \) and \( \beta \) are called the left and right spreads, respectively. Clearly, \( \tilde{A} = (a, \alpha, \beta)_{LR} \) is positive if and only if \( a - \alpha > 0 \) (note that \( L(1) = 0 \)).

Let \( L(x) = R(x) = 1 - x \) then instead \( \tilde{A} = (a, \alpha, \beta)_{LR} \) we simply write \( \tilde{A} = (a, \alpha, \beta) \) and called a LR triangular fuzzy number. Note that we use a fixed function \( L(.) \) and a fixed function \( R(.) \) for all fuzzy numbers in each problem.

Definition 2.3. Two LR fuzzy number \( \tilde{A} = (a, \alpha, \beta) \) and \( \tilde{B} = (b, \gamma, \delta) \) are said to be equal, if and only if \( a = b, \alpha = \gamma \) and \( \beta = \delta \).

Remark 2.1. We consider \( \tilde{0} = (0, 0, 0) \) as a zero LR fuzzy number.

For two fuzzy number, we defined the following operations [9].

Definition 2.4. For two LR fuzzy numbers \( A = (a, \alpha, \beta) \) and \( B = (b, \gamma, \delta) \) the formula for the extended addition becomes:

\[
(a, \alpha, \beta) \oplus (b, \gamma, \delta) = (a + b, \alpha + \gamma, \beta + \delta).
\]

(2.2)

The formula for the extended opposite becomes:

\[
-A = -(a, \alpha, \beta)_{LR} = (-a, \beta, \alpha)_{RL}.
\]

(2.3)

The approximate formula for the extended multiplication of two fuzzy numbers can be summarized as follows:

If \( A > 0 \) and \( B > 0 \), then \( (a, \alpha, \beta) \otimes (b, \gamma, \delta) \simeq (ab, a\gamma + b\alpha, a\delta + b\beta) \).

If \( A > 0 \) and \( B < 0 \), then \( (a, \alpha, \beta) \otimes (b, \gamma, \delta) \simeq (ab, a\gamma - b\beta, a\delta - b\alpha) \).
If $A < 0$ and $B > 0$, then $(a, \alpha, \beta) \otimes (b, \gamma, \delta) \simeq (ab, -a\delta + b\alpha, -a\gamma + b\beta)$. If $A < 0$ and $B < 0$, then $(a, \alpha, \beta) \otimes (b, \gamma, \delta) \simeq (ab, -a\delta - b\beta, -a\gamma - b\alpha)$.  

(2.4) For scalar multiplication:

$$
\lambda \otimes A = \lambda \otimes (a, \alpha, \beta) = \begin{cases} 
(\lambda a, \lambda \alpha, \lambda \beta), & \lambda \geq 0, \\
(\lambda a, -\lambda \beta, -\lambda \alpha), & \lambda < 0.
\end{cases}
$$

(2.5) 

**Definition 2.5.** A matrix $\tilde{A} = (\tilde{a}_{ij})$ is called a fuzzy matrix if each element of $\tilde{A}$ is a fuzzy number [9]. $\tilde{A}$ will be positive (negative) and denoted by $\tilde{A} > 0$ ($\tilde{A} < 0$) if each element of $\tilde{A}$ be positive (negative). Similarly, non-negative and non-positive fuzzy matrices will be defined. We may represent $n \times n$ fuzzy matrix $A = (\tilde{a}_{ij})_{n \times n}$ such that $\tilde{a}_{ij} = (a_{ij}, \alpha_{ij}, \beta_{ij})$, with the new notation $\tilde{A} = (A, M, N)$, where $A = (a_{ij}), M = (\alpha_{ij})$ and $N = (\beta_{ij})$ are three $n \times n$ crisp matrices. Clearly, $A, M$ and $N$ are called the mean value matrix, left and right spread matrices respectively.

3 Dual fully fuzzy linear systems and its solutions

**Definition 3.1.** The $n \times n$ linear system of equations

$$
\begin{cases} 
(\tilde{a}_{11} \otimes \tilde{x}_1) \oplus (\tilde{a}_{12} \otimes \tilde{x}_2) \oplus \cdots \oplus (\tilde{a}_{1n} \otimes \tilde{x}_n) = \tilde{c}_1, \\
(\tilde{a}_{21} \otimes \tilde{x}_1) \oplus (\tilde{a}_{22} \otimes \tilde{x}_2) \oplus \cdots \oplus (\tilde{a}_{2n} \otimes \tilde{x}_n) = \tilde{c}_2, \\
\vdots \\
(\tilde{a}_{n1} \otimes \tilde{x}_1) \oplus (\tilde{a}_{n2} \otimes \tilde{x}_2) \oplus \cdots \oplus (\tilde{a}_{nn} \otimes \tilde{x}_n) = \tilde{c}_n,
\end{cases}
$$

(3.6) 

where $\tilde{a}_{ij}, \tilde{x}_j$ and $\tilde{c}_i, 1 \leq i, j \leq n$ are all fuzzy numbers is called a fully fuzzy linear system (FFLS). The matrix form of this fully fuzzy linear system is $A \otimes \tilde{x} = \tilde{c}$ or simply $\tilde{A} \tilde{x} = \tilde{c}$. Usually there is no inverse element for an arbitrary fuzzy number $\tilde{a} \in E$, i.e. there exists no element $\tilde{b} \in E$ such that $\tilde{a} + \tilde{b} = 0$. Actually, for all non-crisp fuzzy number $\tilde{a} \in E$ we have $\tilde{a} + (-\tilde{a}) \neq 0$. Therefore, the fuzzy system

$$
\begin{cases} 
(\tilde{a}_{11} \otimes \tilde{x}_1) \oplus (\tilde{a}_{12} \otimes \tilde{x}_2) \oplus \cdots \oplus (\tilde{a}_{1n} \otimes \tilde{x}_n) = (\tilde{b}_{11} \otimes \tilde{x}_1) \oplus (\tilde{b}_{12} \otimes \tilde{x}_2) \oplus \cdots \oplus (\tilde{b}_{1n} \otimes \tilde{x}_n) + \tilde{c}_1, \\
(\tilde{a}_{21} \otimes \tilde{x}_1) \oplus (\tilde{a}_{22} \otimes \tilde{x}_2) \oplus \cdots \oplus (\tilde{a}_{2n} \otimes \tilde{x}_n) = (\tilde{b}_{21} \otimes \tilde{x}_1) \oplus (\tilde{b}_{22} \otimes \tilde{x}_2) \oplus \cdots \oplus (\tilde{b}_{2n} \otimes \tilde{x}_n) + \tilde{c}_2, \\
\vdots \\
(\tilde{a}_{n1} \otimes \tilde{x}_1) \oplus (\tilde{a}_{n2} \otimes \tilde{x}_2) \oplus \cdots \oplus (\tilde{a}_{nn} \otimes \tilde{x}_n) = (\tilde{b}_{n1} \otimes \tilde{x}_1) \oplus (\tilde{b}_{n2} \otimes \tilde{x}_2) \oplus \cdots \oplus (\tilde{b}_{nn} \otimes \tilde{x}_n) + \tilde{c}_n,
\end{cases}
$$

(3.7) 

cannot be equivalently replaced by the fuzzy system

$$
(\tilde{A} - \tilde{B})\tilde{x} = \tilde{c},
$$

where the coefficients matrix $\tilde{A} - \tilde{B} = (\tilde{a}_{ij} - \tilde{b}_{ij}), 1 \leq i, j \leq n$ is an $n \times n$ fuzzy matrix, $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)^T$ and $\tilde{c} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n)^T$ are fuzzy number vectors. The system (3.7) is called a dual fully fuzzy linear system (DFFLS). The matrix form of this system is

$$
\tilde{A} \otimes \tilde{x} = \tilde{B} \otimes \tilde{x} + \tilde{c},
$$

(3.8) 

or simply $\tilde{A} \tilde{x} = \tilde{B} \tilde{x} \oplus \tilde{c}$ where the coefficients matrices $\tilde{A} = (\tilde{a}_{ij}) = (A, M, N), \tilde{B} = (\tilde{b}_{ij}) = (B, P, Q) 1 \leq i, j \leq n$ are an $n \times n$ fuzzy matrices, $\tilde{x} = (\tilde{x}_i) = (x, y, z), \tilde{c} = (\tilde{c}_i) = (c, g, h), 1 \leq i, j \leq n$ are an $n \times 1$ fuzzy vectors. So we have

$$
(A, M, N) \otimes (x, y, z) = (B, P, Q) \otimes (x, y, z) \oplus (c, g, h).
$$

Now, it is going to be solved DFFLS by using the approximate multiplication rule for LR fuzzy numbers. For all $1 \leq i, j \leq n, \tilde{a}_{ij} = (a_{ij}, m_{ij}, n_{ij})$ of fuzzy matrix $\tilde{A}$ we define $\tilde{a}^+_{ij} = (a_{ij}^+, m_{ij}^+, n_{ij}^+)$ and $\tilde{a}^-_{ij} = (a_{ij}^-, m_{ij}^-, n_{ij}^-)$ as follows:

$$
\begin{cases} 
\tilde{a}^+_{ij} = (a_{ij}, m_{ij}, n_{ij}) \text{ and } \\
\tilde{a}^-_{ij} = (0, 0, 0) \text{ if } a_{ij} \geq 0,
\end{cases}
$$

(3.9)
This implies $\tilde{a}_{ij} = \tilde{a}_{ij1} \oplus \tilde{a}_{ij2}$ that is $\tilde{a}_{ij}$ is a non-negative mean value and negative mean value respectively. Therefore, fuzzy matrix $\tilde{A} = (\tilde{a}_{ij}) = (A,M,N)$ can be written as $\tilde{A} = A^+ \oplus A^-$ where $A^+ = (\tilde{a}_{ij}) = (A^+,M^+,N^+)$ and $A^- = (\tilde{a}_{ij}) = (A^-,M^-,N^-)$. Clearly, crisp matrix $A^- = (a_{ij})$ is non-positive and residue of crisp matrices $A^+, M^+, N^+, M^-, N^-$ are non-negative. We also have $A = A^+ + A^-, M = M^+ + M^-$ and $N = N^+ + N^-$. Using the same method for the fuzzy matrix $\tilde{B} = (b_{ij}) = (B,P,Q)$ we define $b_{ij} = b_{ij1} \oplus b_{ij2}$ such that $b_{ij} = (b_{ij1},b_{ij2},q_{ij})$ and $\tilde{b}_{ij} = (b_{ij1},p_{ij1},q_{ij})$. Therefore we can write as $B = B^+ + B^-, P = P^+ + P^-$ and $Q = Q^+ + Q^-$. 

Let fuzzy number vector $\tilde{x} = (x,y,z) = (\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_n)^T$ be solution of DFFLS (3.7) where $\tilde{x}_j = (x_{j1},x_{j2},x_{j3})$, $1 \leq j \leq n$, then using the same method we define $\tilde{x}_j = (x^+_j,\tilde{x}_j,\tilde{x}_j)$ and also $y = y^+ + y^-$ and $z = z^+ + z^-$. 

Therefore, in the following theorems, we give an arbitrary solution of DFFLS.

**Theorem 3.1.** A fuzzy number vector $\tilde{x} = (x_1,x_2,\ldots,x_n)^T$ driven by $\tilde{x} = (x,y,z)$, $1 \leq j \leq n$ is called a solution of the DFFLS (3.7) if

$$
\begin{align*}
(A - B)x &= c, \\
(A^+ - B^+)y - (A^- - B^-)z &= g - (M - P)x^+ + (N - Q)x^-, \\
(A^- - B^-)y - (A^+ - B^+)z &= h - (M - Q)x^+ + (N - P)x^-.
\end{align*}

(3.10)
$$

Where $A = A^+ - A^-, B = B^+ - B^-$, $M = M^+ - M^-$, $N = N^+ - N^-$, $P = P^+ - P^-$, $Q = Q^+ - Q^-$, $x = x^+ - x^-$, $y = y^+ - y^-$, and $z = z^+ - z^-$. 

**Proof.** By applying the approximate multiplication rule for $i$-th row of system (3.7), we get

$$
(\tilde{a}_{i1} \oplus \tilde{x}_1) + \cdots + (\tilde{a}_{in} \oplus \tilde{x}_n) = (b_{i1} \oplus \tilde{x}_1) + \cdots + (b_{in} \oplus \tilde{x}_n) + \tilde{c}_i
$$

$$
= (b_{i1}^+ \oplus x^+_1) + \cdots + (b_{in}^+ \oplus x^+_n) + (b_{i1}^- \oplus x^-_1) + \cdots + (b_{in}^- \oplus x^-_n) + c_i
$$

$$
= (a_{i1}^- + a_{i2}^- + \cdots + a_{in}^-) + (a_{i1}^+ + a_{i2}^+ + \cdots + a_{in}^+) + m_{i1}x_1^+ + \cdots + m_{in}x_n^+ + n_{i1}x_1^- + \cdots + n_{in}x_n^-
$$

$$
= (a_{i1}^- + a_{i2}^- + \cdots + a_{in}^-) + (a_{i1}^+ + a_{i2}^+ + \cdots + a_{in}^+) + \tilde{c}_i.
$$

```
\[
\begin{align*}
(\tilde{a}_{i1} \oplus \tilde{x}_1) + \cdots + (\tilde{a}_{in} \oplus \tilde{x}_n) &= (b_{i1} \oplus \tilde{x}_1) + \cdots + (b_{in} \oplus \tilde{x}_n) + \tilde{c}_i, \\

&= (b_{i1}^+ \oplus x^+_1) + \cdots + (b_{in}^+ \oplus x^+_n) + (b_{i1}^- \oplus x^-_1) + \cdots + (b_{in}^- \oplus x^-_n) + c_i, \\

&= (a_{i1}^- + a_{i2}^- + \cdots + a_{in}^-) + (a_{i1}^+ + a_{i2}^+ + \cdots + a_{in}^+) + m_{i1}x_1^+ + \cdots + m_{in}x_n^+ + n_{i1}x_1^- + \cdots + n_{in}x_n^-, \\

&= (a_{i1}^- + a_{i2}^- + \cdots + a_{in}^-) + (a_{i1}^+ + a_{i2}^+ + \cdots + a_{in}^+) + \tilde{c}_i.
\end{align*}
```

where:

$$
\begin{align*}
&\tilde{a}_{ij} = a_{ij1} \oplus a_{ij2}, \\
&\tilde{b}_{ij} = b_{ij1} \oplus b_{ij2} + c_{ij}, \\
&\tilde{c}_i = c_{i1} + \cdots + c_{in}, \\
&b_{ij} = b_{ij1} + b_{ij2} + b_{ij3}, \\
&\tilde{x}_j = x^+_j \oplus x^-_j.
\end{align*}
$$

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Using summation notation, we have

\[
\begin{align*}
\sum_{j=1}^{n} a_{ij} x_j &= \sum_{j=1}^{n} b_{ij} x_j + c_i, \\
\sum_{j=1}^{n} a_{ij}^{+} y_j - \sum_{j=1}^{n} a_{ij}^{-} z_j + \sum_{j=1}^{n} m_{ij}^{+} x_j - \sum_{j=1}^{n} n_{ij} x_j
\end{align*}
\]

\[
= \sum_{j=1}^{n} b_{ij}^{+} y_j - \sum_{j=1}^{n} b_{ij}^{-} z_j + \sum_{j=1}^{n} p_{ij}^{+} x_j - \sum_{j=1}^{n} q_{ij} x_j + g_i, \\
- \sum_{j=1}^{n} a_{ij} y_j + \sum_{j=1}^{n} a_{ij}^{-} z_j + \sum_{j=1}^{n} n_{ij} x_j - \sum_{j=1}^{n} m_{ij} x_j
\]

\[
= - \sum_{j=1}^{n} b_{ij}^{+} y_j + \sum_{j=1}^{n} b_{ij}^{-} z_j + \sum_{j=1}^{n} q_{ij} x_j^{+} - \sum_{j=1}^{n} p_{ij} x_j^{-} + h_i.
\]

Or

\[
\begin{align*}
\sum_{j=1}^{n} (a_{ij} - b_{ij}) x_j &= c_i, \\
\sum_{j=1}^{n} (a_{ij}^{+} - b_{ij}^{+}) y_j - \sum_{j=1}^{n} (a_{ij}^{-} - b_{ij}^{-}) z_j &= g_i - \sum_{j=1}^{n} (m_{ij}^{+} - p_{ij}) x_j^{+} + \sum_{j=1}^{n} (n_{ij} - q_{ij}) x_j^{-}
\end{align*}
\]

(3.11)

By using matrix notation we have

\[
\begin{align*}
(A - B)x &= c, \\
(A^{+} - B^{+})y - (A^{-} - B^{-})z &= g - (M - P)x^{+} + (N - Q)x^{-}, \\
-(A^{-} - B^{-})y + (A^{+} - B^{+})z &= h - \sum_{j=1}^{n} (n_{ij} - q_{ij}) x_j^{+} + \sum_{j=1}^{n} (m_{ij} - p_{ij}) x_j^{-}.
\end{align*}
\]

Therefore, we get the solution of \( FFLS(3.7) \) and the proof is completed.

\[ \square \]

**Corollary 3.1.** Using matrix notation, theorem 3.1 can be written as

\[
\begin{align*}
Rx &= c, \\
SY &= T,
\end{align*}
\]

(3.12)

where matrices \( R, S \) and vectors \( Y, T \) are given as the following:

\[
R = A - B, \quad S = \begin{bmatrix} S_1 & -S_2 \\ -S_2 & S_1 \end{bmatrix}, \quad Y = \begin{bmatrix} y \\ z \end{bmatrix},
\]

(3.13)

where \( x, y, z, x^{+}, x^{-}, c, g, h \) are \( n \times 1 \) crisp vectors, and \( S_1, S_2, M_1, M_2 \) are \( n \times n \) crisp matrices defined as follows:

\[
S_1 = A^{+} - B^{+}, \quad S_2 = A^{-} - B^{-}, \quad M_1 = M - P, \quad M_2 = N - Q.
\]

To clearly mention the order of related matrices (3.12) as follows:

\[
\begin{align*}
R_{n \times n} &= c_{n \times 1}, \\
S_{2n \times 2n} &= Y_{2n \times 1} = Z_{2n \times 1},
\end{align*}
\]
Corollary 3.2. A fuzzy number vector \((\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n)\) driven by \(\vec{x}_j = (x_j, y_j, z_j), 1 \leq j \leq n\) is called a non-negative mean value solution of the DFFLS (3.7) if

\[
\begin{align*}
(A - B)x &= c, \\
(A^+ - B^-)y - (A^- - B^-)z &= g - (M - P)x, \\
-(A^- - B^-)y + (A^+ - B^+)z &= h - (N - Q)x.
\end{align*}
\]

Where \(A = A^+ + A^-, \ B = B^+ + B^-\). Clearly that we can be write vector \(Z\) in (3.13) as follows:

\[
Z = \begin{bmatrix} g \\ h \end{bmatrix} - \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} x
\]

Corollary 3.3. A fuzzy number vector \((\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n)\) driven by \(\vec{x}_j = (x_j, y_j, z_j), 1 \leq j \leq n\) is called a negative mean value solution of the DFFLS (3.7) if

\[
\begin{align*}
(A - B)x &= c, \\
(A^+ - B^-)y - (A^- - B^-)z &= g + (N - Q)x, \\
-(A^- - B^-)y + (A^+ - B^+)z &= h + (M - P)x.
\end{align*}
\]

Where \(A = A^+ + A^-, \ B = B^+ + B^-\). Clearly that we can be write vector \(Z\) in (3.13) as follows:

\[
Z = \begin{bmatrix} g \\ h \end{bmatrix} + \begin{bmatrix} M_2 \\ M_1 \end{bmatrix} x
\]

Theorem 3.2. The matrix \(S\) is non-singular, if and only if the matrices \(S_1 + S_2\) and \(S_1 - S_2\) are both non-singular. (see [10])

Theorem 3.3. If \(S^{-1}\) exists, it must have the same structure as \(S\) (see [2, 10]), i.e.:

\[
S^{-1} = \begin{bmatrix} D & E \\ E & D \end{bmatrix}
\]

where

\[
\begin{align*}
D &= \frac{1}{2}[(S_1 + S_2)^{-1} + (S_1 - S_2)^{-1}], \\
E &= \frac{1}{2}[(S_1 + S_2)^{-1} - (S_1 - S_2)^{-1}].
\end{align*}
\]

Therefore, obtaining \(x, y, z\) via equations systems DFFLS (3.7) may be left and right spreads \(y, z\) yield to negative thus in the following definition we define fuzzy solution.

Definition 3.2. Let \(\vec{x} = (x_j, y_j, z_j), 1 \leq j \leq n\) denote the unique solution of Eq. (3.15). The fuzzy number vector \(\vec{u} = (u_j, v_j, w_j), 1 \leq j \leq n\) defined by

\[
\begin{align*}
u_j &= x_j, 1 \leq j \leq n \\
v_j &= |y_j|, 1 \leq j \leq n \\
w_j &= |z_j|, 1 \leq j \leq n,
\end{align*}
\]

is called the fuzzy solution of Eq. (3.15).

On the other hand if \((x_j, y_j, z_j), 1 \leq j \leq n\) are all fuzzy numbers (i.e. \(y_j, z_j \geq 0\)) then \(v_j = y_j, w_j = z_j, 1 \leq j \leq n\); otherwise \(v_j = |y_j|, w_j = |z_j|, 1 \leq j \leq n\) and \(u\) is called a fuzzy solution.

4 Numerical examples

In this section, we take some numerical example.
Example 4.1. Consider the following DFFLS

\[
\{ (3,1,2) \otimes \bar{x}_1 \oplus (1,1,3) \otimes \bar{x}_2 = (1,1,2) \otimes \bar{x}_1 \oplus (2,3,1) \otimes \bar{x}_2 + (2,2,1), \\
(−2,2,1) \otimes \bar{x}_1 \oplus (4,1,1) \otimes \bar{x}_2 = (3,1,3) \otimes \bar{x}_1 \oplus (1,5,2) \otimes \bar{x}_2 + (−4,3,2).
\]

To find the solution of this system, we solve two crisp linear systems (3.12) (i.e. \(R \mathbf{x} = \mathbf{c}, \ \mathbf{S} \mathbf{Y} = \mathbf{Z}\), where:

\[
R = \begin{bmatrix}
2 & -1 & 0 \\
3 & 3 & 2 \\
0 & 2 & -1 \\
2 & -4 & 3
\end{bmatrix}, \ S = \begin{bmatrix}
2 & 0 & 0 \\
0 & 3 & 2 \\
1 & 2 & 0 \\
0 & 0 & 1
\end{bmatrix}, \ c = \begin{bmatrix}
6 \\
9 \\
3
\end{bmatrix}, \ Z = \begin{bmatrix}
4.4667 \\
2.9333 \\
3.4000
\end{bmatrix}.
\]

By using the inverse of matrices \(R\) and \(S\), we have:

\[
x = R^{-1}c = \begin{bmatrix}
2 \\
2
\end{bmatrix}, \ Y = S^{-1}Z = \begin{bmatrix}
0.2 & 0.3 & 0.1 & 0.2 \\
0.3 & 0.1 & 0.2 & 0.4 \\
0.3 & 0.1 & 0.2 & 0.1 \\
0.2 & 0.3 & 0.4 & 0.1
\end{bmatrix}, \ 0.2 & 0.3 & 0.1 & 0.2 \\
0.4 & 0.2 & 0.1 & 0.3 \\
0.3 & 0.2 & 0.1 & 0.4 \\
0.5 & 0.1 & 0.6 & 0.3
\end{bmatrix}, \ Q = \begin{bmatrix}
0.2 & 0.4 & 0.1 & 0.5 \\
0.2 & 0.4 & 0.1 & 0.6 \\
0.2 & 0.4 & 0.1 & 0.5 \\
0.2 & 0.4 & 0.1 & 0.6
\end{bmatrix}.
\]

Therefore, the solution of this DFFLS is \(\bar{x} = (\bar{x}_1, \bar{x}_2)^T = ((2,4.4667,0.2000), (2,2.9333,3.4000))\).

Example 4.2. Let \(\tilde{\mathbf{A}} = (A,M,N), \tilde{\mathbf{B}} = (B,P,Q)\) be fully fuzzy matrices and \(\tilde{\mathbf{c}} = (c,g,h)\) be a fully fuzzy vector, with

\[
A = \begin{bmatrix}
5 & 2 & 3 & 4 \\
-3 & 6 & 2 & -1 \\
4 & 2 & 8 & 6 \\
2 & 1 & -1 & 0
\end{bmatrix}, \ M = \begin{bmatrix}
0.1 & 0.3 & 0.2 & 0.5 \\
0.1 & 0.2 & 0.4 & 0.1 \\
0.2 & 0.3 & 0.5 & 0.1 \\
0.1 & 0.2 & 0.3 & 0.4
\end{bmatrix}, \ N = \begin{bmatrix}
0.2 & 0.3 & 0.1 & 0.2 \\
0.3 & 0.1 & 0.2 & 0.4 \\
0.3 & 0.1 & 0.2 & 0.1 \\
0.2 & 0.3 & 0.1 & 0.4
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
2 & 0 & -2 & -4 \\
-1 & 2 & 1 & 2 \\
3 & -1 & 4 & 1 \\
1 & 2 & -7 & -2
\end{bmatrix}, \ P = \begin{bmatrix}
0.2 & 0.1 & 0.4 & 0.3 \\
0.5 & 0.4 & 0.1 & 0.2 \\
0.1 & 0.2 & 0.4 & 0.6 \\
0.2 & 0.4 & 0.1 & 0.5
\end{bmatrix}, \ Q = \begin{bmatrix}
0.5 & 0.1 & 0.4 & 0.2 \\
0.4 & 0.2 & 0.1 & 0.3 \\
0.3 & 0.2 & 0.1 & 0.4 \\
0.5 & 0.1 & 0.6 & 0.3
\end{bmatrix},
\]

\[
c = \begin{bmatrix}
3 & 6 & 2 & 8
\end{bmatrix}^T, \ g = \begin{bmatrix}
0.8 & 0.9 & 0.4 & 0.7
\end{bmatrix}^T, \ h = \begin{bmatrix}
0.9 & 0.7 & 0.8 & 0.4
\end{bmatrix}^T.
\]

Then we define the solution of systems (3.8) (i.e. \(\bar{\mathbf{A}} \otimes \bar{x} = \bar{\mathbf{B}} \otimes \bar{x} \oplus \bar{\mathbf{c}}\)).

To find the solution of the DFFLS, we solve two crisp linear systems (3.12) (i.e. \(R \mathbf{x} = \mathbf{c}, \ \mathbf{S} \mathbf{Y} = \mathbf{Z}\), where:

\[
R = \begin{bmatrix}
3 & 2 & 5 & 8 \\
-2 & 4 & 1 & -3 \\
1 & 3 & 4 & 5 \\
1 & -1 & 6 & 4
\end{bmatrix}, \ S = \begin{bmatrix}
3 & 2 & 3 & 4 & 0 & 0 & 0 & -2 & -4 \\
1 & 2 & 4 & 5 & 0 & -1 & 0 & 0 \\
0 & 0 & -2 & -4 & 3 & 2 & 3 & 4 \\
2 & 0 & 0 & 1 & 0 & 4 & 1 & -2
\end{bmatrix}, \ Z = \begin{bmatrix}
1.5914 \\
1.3743 \\
0.5543 \\
0.3543 \\
1.2527 \\
1.1286 \\
1.9857 \\
3.0028
\end{bmatrix}.
\]

By using the inverse of matrices \(R\) and \(S\), we have:

\[
x = R^{-1}c = \begin{bmatrix}
3.6286 \\
0.5429 \\
2.6857 \\
-2.8000
\end{bmatrix}, \ Y = \begin{bmatrix}
y \\
z
\end{bmatrix} = S^{-1}Z = \begin{bmatrix}
1.6888 \\
0.5145 \\
-0.5438 \\
-0.0286 \\
-0.4713 \\
-0.1544 \\
-0.1118 \\
0.7456
\end{bmatrix}.
\]
Since some components of $y, z$ (left and right spreads) in this example are negative, then we have a fuzzy solution. So by taking:

$$u_j = x_j, \quad v_j = |y_j|, \quad w_j = |z_j|, \quad \text{for} \quad 1 \leq j \leq 4,$$

the fuzzy solution $\bar{u} = (u_j, v_j, w_j), 1 \leq j \leq n$ is obtained and we have

$$\bar{u} = \begin{pmatrix}
(3.6286, 1.6888, 0.4713) \\
(0.5429, 0.5145, 0.1544) \\
(2.6857, 0.5438, 0.1118) \\
(-2.8000, 0.0286, 0.7456)
\end{pmatrix}.$$

5 Conclusion

In this paper a solution of dual fully fuzzy linear system (DFFLS) was found. Indeed, a method was introduced, where the obtained solution is a LR fuzzy number vector. The mean value vector was obtained from the solution of the $n \times n$ crisp linear system $Ax = b$. By solving this system, three solutions were obtained where they may be negative, non-negative or a combination of both. To finding the spreads of fuzzy number also, matrix equation $SY = Z$ was solved, where $S$ is a $2n \times 2n$ matrix.

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