Analyzing buffer contents of space division output buffered multichannel switches

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Abstract
In digital communication systems, buffers are used for the temporary storage of digital information awaiting transmission via some communication channel(s). In this article, buffer analysis of space division output buffered switches operating in an ATM environment is presented. The switch is modeled as a discrete-time, batch arrival, multiserver queuing system, with infinite buffer and geometric service time. Probability Generating Functions (PGFs) of the system occupancy are derived for single server, two servers, and a general form for multiserver systems. Expectations for the corresponding random variables are obtained for different cases. Many numerical examples, which verify the results, are presented.

Keywords: Data Communications; Performance Analysis; Queuing Analysis; Output Buffered Switches; Multiserver Queues; System Occupancy.

1 Introduction

Recently, with the advent of ATM-based multiservice networks, a renewed interest in discrete-time models becomes apparent. Usually, the quantities studied in these investigations are the buffer occupancy, system occupancy, and the delays (waiting times) experienced by the packets in the buffer [1], [2]. In [8], [9] the delays in various multiserver systems are investigated, and in [3] packet delays in single server systems with various types of first order Markovian correlated arrival processes are studied. Also, the discrete-time queues with or without server interruptions have received great attention in the scientific literature. Both single server [10] and multiserver [5] have been analyzed. One of the earlier papers is the analysis by [11], who investigated a finite multiserver queue without server interruptions. Most authors, however, analyze an infinite system. Some make specific assumptions about the arrival process [12] while general independent arrivals are considered elsewhere [13]. Several models have been used to describe the interruption process. While both analytical and numerical results have been obtained on many occasions with respect to performance measures related to the buffer contents distribution [4],[3] for the case of multiserver [5], [6] as well as for the single server case [7] the derivation of delay characteristics has received much less attention in the past. More specifically, analytic results have been limited primarily to the single server case, whereas the multiple server case has mainly been investigated by means of numerical methods. In a number of papers, delays are analyzed for single server queueing systems where packets arrive according to an correlated arrival process. The rest of this
paper is organized as follows. Mathematical model assumptions are presented in the next section. In Section 3 we analyze the system occupancy and derive PGFs of system occupancy for single server, two servers and general formulas for multiserver system. Expectations and numerical examples are given in section 4. Section 5 is for conclusions.

## 2 Model Assumptions

![Diagram of an output buffered space division switch](image)

Figure 1: An $N \times N$ output buffered space division switch.

As largely reflected by Figure 1, there are $N$ input ports and $N$ buffered output ports each with $c$ channels. Every slot a packet will arrive with probability $r$ or will not arrive with probability $\overline{r} = 1 - r$. The probability that the packet requests a particular output port $i$ is $\frac{1}{cN}$, for all $i = 1, 2, \cdots, N$. As indicated in Figure 2, in each slot $k$, $A^k = 0, 1, \cdots, N$, packets arrive from the input ports into the port. The $A^k$ are independent and identically distributed (iid) RVs with PGF $A(z)$. That is

\begin{equation}
A(z) = \left(1 - \frac{r}{cN} + \frac{r}{cN}z\right)^N.
\end{equation}

\begin{equation}
A'(1) = \frac{r}{c}, \quad A''(1) = \left(\frac{r}{c}\right)^2 \left(\frac{N-1}{N}\right).
\end{equation}

In view of Figure 2, let $D^{k+1}$ be the number of packets that will leave the port at the end of slot $k+1$ with distribution $d_i$. In each slot a packet leaves a server with probability $s$ or does not leave with probability $\overline{s}$ then the number of departing packets per slot follows a binomial distribution. This implies that the service time of packets are geometrically distributed with parameters $s$. Let $X^k = 1, 2, \ldots$, be the service time of the packet that arrives into the port in slot $k$. It is clear that the $X^k$ are iid. Let $x_i$ and $X(z)$ be the common distribution and common PGF of $X^k$. From the assumptions, it can be shown that $x_i = s^i - 1$, and that

\begin{equation}
X(z) = \frac{s^x}{1 - \overline{s}z}.
\end{equation}
3 System Occupancy

Let $P_k = 0, 1, \cdots$, be a RV denoting the port occupancy in slot $k$, with distribution $p_k^j$ and PGF $P_k(z)$. That is

$$P_k(z) = \sum_{j=0}^{\infty} p_k^j z^j = E \left[ z^j \right]. \quad (3.4)$$

Looking at the existing packets $P_k$ at slot $k$ as independent trials where in each slot a packet leave a server with probability $s$ or does not leave with probability $\bar{s}$, then the number of packets served by the end of slot $k + 1 (D_{k+1})$ is going to depend on $P_k$, with the following conditional distribution

$$Pr \left[ D_{k+1} = j \mid P_k = i \right]$$

$$= \begin{cases} 
(j \choose i) s^i \bar{s}^{j-i} & \text{if } j < c, i \leq j \\
(c \choose i) s^j \bar{s}^{c-j} & \text{if } j \geq c, i \leq c \\
0 & \text{otherwise}
\end{cases} \quad (3.5)$$

The evaluation of the system occupancy in two successive slots $k, k + 1$ can be described, in view of figure (2), by the following RV equation

$$P_{k+1} = P_k - D_{k+1} + A_{k+1}. \quad (3.7)$$

Using (3.7) and all the possible combinations of $P_k, D_{k+1}$ in (3.4), then applying (3.5), and after some manipulation, we get

$$P_{k+1}(z) = E \left[ z^{P_k - D_{k+1} + A_{k+1}} \right] = E \left[ z^{A_{k+1}} \right] z^{P_k - D_{k+1}}$$

$$= A(z) \sum_{j=0}^{c-1} \left( (s + \bar{s} z)^j - (s + \bar{s} z)^c z^{j-c} \right) p_j^k + A(z) (s + \bar{s} z)^c z^{-c} P_k(z). \quad (3.8)$$

At the steady state,

$$P(z) = \frac{A(z) z^c}{z^c - A(z) (s + \bar{s} z)^c} \sum_{j=0}^{c-1} \left( (s + \bar{s} z)^j - (s + \bar{s} z)^c z^{j-c} \right) p_j. \quad (3.9)$$

To define the PGF $P(z)$, given by (3.9) completely, we have to define the unknown probabilities $p_j, j = 0, 1, \ldots, c - 1$. We will apply Rouche’s Theorem to the denominator of (3.9), and proceed to consider points $\xi_m$, within the unit disk for which the denominator of (3.9) is equal to zero. Then

$$\xi_m^c = A(\xi_m^c) (s + \bar{s} \xi_m^c)^c. \quad (3.10)$$
For any such point $\xi_n$, $|\xi_n| \leq 1, n = 0, 1, 2, \ldots, c − 1$ we must have a simple zero. It can be shown that the function $P(z)$ is bounded within the unit disk $|z| \leq 1$ therefore, both the numerator and the denominator of (3.9) must be zero for the same values of $z$. Then substituting with the zeros $\xi_n$ in the numerator of (3.9) and provided that $A(\xi_n) \neq 0$, we get
\[
\xi_n^{c} \sum_{j=0}^{c-1} \left[ (s + \xi_n)^j - (s + \xi_n)^c \xi_n^{j-c} \right] p_j = 0, \quad n = 1, 2, \ldots, c - 1,
\] (3.11)
which are $c - 1$ equations in $c$ unknowns. The equation number $c$ needed to solve for the unknown probabilities comes from the normalization condition $P(1) = 1$ but let us first write (3.9), in the form
\[
P(z) = \frac{A(z)\Phi(z)}{z^{c} - A(z)(s + z)^{c}}, \quad (3.12)
\]
\[
\Phi(z) = z^{c} \sum_{j=0}^{c-1} \left[ (s + z)^j - (s + z)^c z^{j-c} \right] p_j,
\]
\[
\Phi(1) = 0. \quad (3.13)
\]
Taking the first derivative of (3.13) at $z = 1$, thus
\[
\Phi'(1) = s \sum_{j=0}^{c-1} [c - j] p_j. \quad (3.14)
\]
Applying the normalization condition to (3.12) after applying L'Hospital's rule, we get
\[
\Phi'(1) = sc - \frac{r}{c}. \quad (3.15)
\]
where $A'(1) = \xi$. Now, equating (3.14) and (3.15), hence
\[
s \sum_{j=0}^{c-1} [c - j] p_j = sc - \frac{r}{c}. \quad (3.16)
\]
Equations (3.11) and (3.16) can be written explicitly, as the following $c$ equations
\[
[\xi_1^{c} - (s + \xi_1)^c] p_0 + [\xi_1^{c} (s + \xi_1) - (s + \xi_1)^c \xi_1^{c-1}] p_1 + \cdots + [\xi_1^{c} (s + \xi_1)^{c-1} - (s + \xi_1)^c \xi_1^{c-1}] p_{c-1} = 0
\]
\[
[\xi_2^{c} - (s + \xi_2)^c] p_0 + [\xi_2^{c} (s + \xi_2) - (s + \xi_2)^c \xi_2^{c-1}] p_1 + \cdots + [\xi_2^{c} (s + \xi_2)^{c-1} - (s + \xi_2)^c \xi_2^{c-1}] p_{c-1} = 0
\]
\[
\vdots
\]
\[
[\xi_{c-1}^{c} - (s + \xi_{c-1})^{c}] p_0 + [\xi_{c-1}^{c} (s + \xi_{c-1}) - (s + \xi_{c-1})^c \xi_{c-1}^{c-1}] p_1 + \cdots + [\xi_{c-1}^{c} (s + \xi_{c-1})^{c-1} - (s + \xi_{c-1})^c \xi_{c-1}^{c-1}] p_{c-1} = 0
\]
\[
scp_0 + s[c - 1] p_1 + \cdots + p_{c-1} = sc - \frac{r}{c}.
\] (3.17)
which are $c$ equations in the $c$ unknowns $p_j$, $j = 0, 1, \ldots, c - 1$.
and can be solved numerically to find the unknown probabilities, as soon as the value of $c$ is specified. Now, to continue our calculation and measurement we have to define the value of $c$, by taking some special cases for $c$. 


Case 1: PGF of the System Occupancy When $c = 1$. Substituting for $c = 1$ in (3.9), thus

$$P(z) = \frac{A(z) \left[ z - (s + \bar{c}z) \right] p_0}{z - A(z) (s + \bar{c}z)}.$$  
(3.18)

After finding $p_0$

$$P(z) = \left( 1 - \frac{r}{s} \right) \frac{A(z) \left[ z - (s + \bar{c}z) \right]}{z - A(z) (s + \bar{c}z)}.$$  
(3.19)

Case 2: PGF of the System Occupancy When $c = 2$. Substituting for $c = 2$ in (3.9), thus

$$P(z) = \frac{1}{z^2 - A(z) (s + \bar{c}z)^2} \left( A(z) \left[ z^2 - (s + \bar{c}z)^2 \right] p_0 \right)$$
$$+ A(z) \left[ z^2 (s + \bar{c}z) - (s + \bar{c}z)^2 z \right] p_1.$$  
(3.20)

After calculating $p_0$ and $p_1$, we get

$$P(z) = \frac{A(z)}{z^2 - A(z) (s + \bar{c}z)^2} \left\{ \left[ z^2 - (s + \bar{c}z)^2 \right] \left( -\frac{\beta_1 (2s - r)}{\alpha_1 \beta_2 - \alpha_2 \beta_1} \right) \right\}$$
$$+ \left[ z^2 (s + \bar{c}z) - (s + \bar{c}z)^2 z \right] \left( \frac{\alpha_1 (2s - r)}{\alpha_1 \beta_2 - \alpha_2 \beta_1} \right),$$  
(3.21)

$$\alpha_1 = \left[ \frac{\xi_1^2 - (s + \bar{c}\xi_1)^2}{s} \right], \quad \alpha_2 = 2s$$
$$\beta_1 = \left[ \frac{\xi_1^2 (s + \bar{c}\xi_1) - (s + \bar{c}\xi_1)^2 \xi_1}{s} \right], \quad \beta_2 = s.$$  
(3.22)

4 Expectations and Numerical Examples

4.1 Expected Value of the System Occupancy When $c = 1$

To ease the calculation of the expected value of the system occupancy let us write (3.19), in the form

$$P(z) = \left( 1 - \frac{r}{s} \right) \frac{\Phi(z)}{\theta(z)}.$$  
(4.23)

$$\Phi(z) = A(z) \left[ z - (s + \bar{c}z) \right],$$  
(4.24)

$$\theta(z) = z - A(z) (s + \bar{c}z).$$  
(4.25)

Taking the first derivative of (4.23) at $z = 1$, using L’Hospital’s rule twice, and after some manipulation, we will get

$$E[P] = \frac{A'' (1) + 2r - 2r^2}{2(s - r)},$$  
(4.26)

$$= \frac{1}{2(s - r)} \left\{ r^2 \left( \frac{N - 1}{N} + 2r - 2r^2 \right) \right\}.$$
4.2 Expected Value of the System Occupancy When \( c = 2 \)

Taking the first derivative of (3.21) at \( z = 1 \), we can show that

\[
E[P] = \frac{1}{(4s-r)} \left[ 2s[2-s] (\sigma_1 + \sigma_2) + rs (2\sigma_1 + \sigma_2) \right] - \frac{(2\sigma_1 + \sigma_2)s}{(4s-r)^2} \times \]

\[
\frac{\left[ 4s[2-s] - 4r\bar{s} - 2\bar{s}'' (1) \right]}{(4s-r)} - \frac{(2\sigma_1 + \sigma_2)s}{(4s-r)^2} \left[ 4s[2-s] - 4r\bar{s} - \frac{N-1}{2} \frac{N}{2} \right],
\]

where \( \sigma_1, \sigma_2 \) are given by

\[
\sigma_1 = -\frac{\beta_1(4s-r)}{2(\alpha_1\beta_2 - \alpha_2\beta_1)},
\]

\[
\sigma_2 = \frac{\alpha_1(4s-r)}{2(\alpha_1\beta_2 - \alpha_2\beta_1)}.
\]

Figure 3: Expected system occupancy \( E[P] \) vs the arrival rate \( r; N = 8; c = 1 \).

Now we introduce many numerical examples which verify the results. From figure (3) we note that, the expected value of the system occupancy for \( c = 1 \) increases when increasing the arrival rate of the packets for a given values of \( N, s \).

Again, we will focus on the expected value of the system occupancy but for \( c = 2 \). In figure (5), the expected value of the system occupancy is plotted as a function in the service rate for arrival rate \( r = 0.1, 0.2, 0.3 \), and \( N = 8 \). As clear in the figure, the expected value of the system occupancy increases when the service rate decreases for a given arrival rate. Moreover, as the arrival rate increases and the service rate increases, this gives the best \( E[P] \).
Figure 4: Expected system occupancy $E[P]$ vs the service rate $s, N = 8, c = 1$.

Figure 5: Expected system occupancy $E[P]$ vs the service rate $s, N = 8, c = 2$. 
Figure 6: Expected system occupancy $E[P]$ vs the arrival rate $r, N = 8, s = 0.9$.

Figure 7: Expected system occupancy $E[P]$ vs the service rate $s, N = 8, r = 0.1$. 
5 Conclusion

In this article we present buffer analysis of space division output buffered switches operating in an ATM environment. The main contributions have been using multiserver case and geometric service time. The switch is modeled as a discrete-time, batch arrival, multiserver queuing system, with infinite buffer and geometric service times. We have obtained PGFs of system occupancy for single server, two servers and general form for multiserver systems. The PGFs have been used to derive the corresponding expectations for different cases. The results of the analysis have been verified by many numerical examples.

References


