Convergence and stability of the incomplete factorizations of H-Matrices

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Abstract
The purpose of this paper is to show that the incomplete LU factorizations of the H-matrices by using the incomplete LU-factorizations of an H-matrix are at least as stable as the complete LU-factorizations of its comparison matrix. We give also some new characterizations of the H-matrices in connection with their incomplete LU-factorizations.

Key words: Stability; Mild stability; Gaussian elimination; Incomplete LU–factorization.

1 Introduction
Gaussian Elimination for Solving \( Ax = b \).

Consider the problem of solving the linear system of \( n \) equations in \( n \) unknowns:

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
    \vdots \\
    a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{align*}
\]

or, in matrix notation, \( Ax = b \) where \( A = (a_{ij}) \) and \( b = (b_1, \ldots, b_n)^T \).

A well-known approach to solving the problem is the classical elimination scheme known as Gaussian elimination. The basic idea is to reduce the system to an equivalent upper triangular system so that reduced upper triangular system can be solved easily using the back-substitution algorithm. The reduction process consists of \( n - 1 \) steps. In the following,

\[
b^{(k)} = \begin{pmatrix} b_1^{(k)} \\ b_2^{(k)} \\ \vdots \\ b_n^{(k)} \end{pmatrix} \quad A^{(k)} = \begin{pmatrix} a_{11}^{(k)} \\ a_{21}^{(k)} \\ \vdots \\ a_{n1}^{(k)} \end{pmatrix}, \quad A^{(0)} = A, \quad b^{(0)} = b
\]

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Step 1.
At step 1, the unknown x1 is eliminated from the second through the nth equations. This is done by multiplying the first equation by
\[
\begin{pmatrix}
-a_{21} \\
a_{11}
\end{pmatrix}
\begin{pmatrix}
-a_{31} \\
a_{11}
\end{pmatrix}
\ldots
\begin{pmatrix}
-a_{n1} \\
a_{11}
\end{pmatrix}
\]
and adding it to the second through nth equations. The quantities \( m_i = -\frac{a_{1i}}{a_{11}} \) (i = 2, ..., n) are called multipliers. At the end of step 1, the system \( Ax = b \) becomes \( A^{(1)}x = b^{(1)} \), where the entries of \( A^{(1)} = (a^{(1)}_{ij}) \) and those of \( b^{(1)} \) are related to the entries of \( A \) and \( b \) as follows:
\[
a^{(1)}_{ij} = a_{ij} + m_i a_{1j} \quad (i = 2, ..., n; j = 2, ..., n)
\]
\[
b^{(1)}_i = b_i + m_i b_{1j} \quad (i = 2, ..., n)
\]
(Note that \( a^{(1)}_{21}, a^{(1)}_{31}, \ldots, a^{(1)}_{n1} \) are all zero.)

Step 2.
At step 2, x2 is eliminated from the third through the nth equations of \( A^{(1)}x = b^{(1)} \) by multiplying the second equations of \( A^{(1)}x = b^{(1)} \) by the multipliers \( m_2 = -\frac{a_{22}^{(1)}}{a_{22}} \), i = 3, ..., n and adding the result to the third through nth equations. The system now becomes:
\[
A^{(2)}x = b^{(2)}, \quad \text{whose entries are given as follows:}
\]
\[
a^{(2)}_{ij} = a^{(1)}_{ij} + m_j a_{2j}^{(1)} \quad (i = 3, ..., n; j = 3, ..., n)
\]
\[
b^{(2)}_i = b^{(1)}_i + m_j b_{2j}^{(1)} \quad (i = 3, ..., n)
\]
\[
b^{(2)}_2 = 0 \quad (i = 3, ..., n)
\]
The other entries of \( A^{(2)} \) and those of \( b^{(2)} \) are the same as those of \( A^{(1)} \) and \( b^{(1)} \) respectively.

Step k.
At step k, the \((n - k)\) multipliers \( m_k = -\frac{a^{(k-1)}_{kk}}{a^{(k-1)}_{kk}} \), i = k + 1, ..., n are formed; using them, xk is eliminated from the \((k + 1)\)st through the nth equations of \( A^{(k-1)}x = b^{(k-1)} \). The entries of \( A^{(k)} \) and those of \( b^{(k)} \) are given by
\[
a^{(k)}_{ij} = a^{(k-1)}_{ij} + m_k a^{(k-1)}_{ij} \quad (i = k + 1, ..., n; j = k + 1, ..., n)
\]
\[
b^{(k)}_i = b^{(k-1)}_i + m_k b^{(k-1)}_i \quad (i = k + 1, ..., n)
\]
\[
a^{(k)}_{ik} = 0 \quad (i = k + 1, ..., n)
\]
The other entries of \( A^{(k)} \) and \( b^{(k)} \) are the same as those of \( A^{(k-1)} \) and \( b^{(k-1)} \) respectively.

Step n – 1.
At the end of the \((n - 1)\)st step, the reduced matrix \( A^{(n-1)} \) is upper triangular and the original vector \( b \) is transformed to \( b^{(n-1)} \).

Definitions and concepts of stability
Definition 1.
An algorithm Will be called forward stable if the computed solution \( \hat{x} \) is close to the exact, \( x \), in some sense.
Definition 2.
An algorithm is called backward stable if it produces an exact solution to a nearby problem.

Definition 3.
An algorithm for solving $Ax = b$ will be called stable if the computed solution $\hat{x}$ is such that $(A + E)\hat{x} = b - \delta b$ with $E$ and $\delta b$ small.

Example 1.
The Gaussian elimination algorithm without row changes is unstable for arbitrary matrices.

Definition 4.
An algorithm is stable for a class of matrices $C$ if for every matrix $A$ in $C$, the computed solution by the algorithm is the exact solution of a nearby problem.

Thus, for the Linear system problem $Ax = b$ an algorithm is stable for a class of matrices $C$ if every $A \in C$ and for each $b$, it produces a computed solution $\hat{x}$ that satisfies $(A+E)\hat{x} = b - \delta b$ for some $E$ and $\delta b$, where $(A + E)$ is close to $A$ and $b - \delta b$ is close to $b$.

Strong, Weak, and Mild Stability

Definition 5.
An algorithm for solving the linear system problem $Ax = b$ is strongly stable for a class of matrices $C$ if, for each $A$ in $C$ the computed solution is the exact solution of a nearby problem, and the matrix $(A + E)$ also belongs to $C$.

Example 2.
The Gaussian elimination with pivoting is strongly stable on the class of nonsingular matrices.

Definition 6.
An algorithm is weakly stable for a class of matrices $C$ if for each well-conditioned matrix in $C$ the algorithm produces an acceptable accurate solution.

Definition 7.
An algorithm is mildly stable if it produces a solution that is close to the exact solution of a nearby problem.

Gaussian Elimination without Pivoting

Set $A = A^{(0)}$.

Step 1.
Find an elementary matrix $E_1$ such that $A^{(1)} = E_1A$ has zeros below the $(1,1)$ entry in the first column. That is, $A^{(1)}$ has the form

$$A^{(1)} = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\
\vdots & \vdots & \ddots & \vdots \\
0 & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)}
\end{pmatrix}$$
Note that it is sufficient only to find $E_1$ such that

$$
E_1 = \begin{pmatrix}
    a_{12} & \cdots & a_{1n} \\
    a_{22} & \cdots & a_{2n} \\
    \vdots & \ddots & \vdots \\
    a_{k2} & \cdots & a_{kn}
\end{pmatrix}
$$

Then $A^{(1)} = E_1A$ will have the preceding from and is the same as the matrix $A^{(1)}$ obtained at the end of step 1 of Algorithm. Record the multipliers: $m_{21}, m_{31}, \ldots, m_{n1}, \ldots, m_{n1} = \frac{-a_{ii}}{a_{11}}, i = 2, \ldots, n$.

**Step 2.**

Find an elementary matrix $E_2$ such that $A^{(2)} = E_2A^{(1)}$ has zeros below the (2,2) entry in the second column. The matrix $E_2$ can be constructed as follows.

First, find an elementary matrix $\hat{E}_2$ of order $(n - 1)$ such that

$$
\hat{E}_2 = \begin{pmatrix}
    1 & 0 & \cdots & 0 \\
    0 & \cdots & \cdots & a_{(n-1)n} \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & \cdots & 0 & 1
\end{pmatrix}
$$

Record the multipliers: $m_{32}, \ldots, m_{n2}$; $m_{i2} = -a_{i2}^{(1)} / a_{22}^{(1)}$, $i = 3, \ldots, n$. Then define

$$
E_2 = \begin{pmatrix}
    1 & 0 & \cdots & 0 \\
    0 & \cdots & \cdots & a_{1n} \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & \cdots & 0 & 1
\end{pmatrix} = \begin{pmatrix}
    1 & 0 \\
    0 & \hat{E}_2
\end{pmatrix}
$$

Then $A^{(2)} = E_2A^{(1)}$ will then have zeros below the (2,2) entry in the second column.

$$
A^{(2)} = \begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    0 & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & a_{nn}
\end{pmatrix}
$$

Note that premultiplication of $A^{(1)}$ by $E_2$ does not does not destroy zeros already created in $A^{(1)}$. This matrix $A^{(2)}$ is the same as the matrix $A^{(2)}$ of Algorithm.

**Step k.**

In general, at the kth step, an elementary matrix $E_k$ is found such that $A^{(k)} = E_kA^{(k-1)}$ has zeros below the $(k,k)$ entry in the kth column. $E_k$ is computed in two successive steps. First, an elementary matrix $\hat{E}_k$ of order $n - k + 1$ is constructed such that

$$
\hat{E}_k = \begin{pmatrix}
    a_{k2}^{(k-1)} \\
    a_{k3}^{(k-1)} \\
    \vdots \\
    a_{kn}^{(k-1)}
\end{pmatrix}
$$

and then $E_k$ is defined as $E_k = \begin{pmatrix} I_{k-1} & \hat{E}_k \\ \hat{E}_k & 0 \end{pmatrix}$.

Here $I_{k-1}$ is the matrix of the first $(k - 1)$ rows and columns of the $n \times n$ identity matrix $I$. The matrix $A^{(k)} = E_kA^{(k-1)}$ is the same as the matrix $A^{(k)}$ of Algorithm (*).
Define the multipliers: $m_{k+1,k}, \ldots, m_{k,n}$; 

$$m_{i,k} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k-1)}}, \quad i = k + 1, \ldots, n$$

**Step n – 1.**

At the end of the $(n – 1)$st step, the matrix $A^{(n-1)}$ is upper triangular and the same matrix $A^{(n-1)}$ of Algorithm.

$$A^{(n-1)} = \begin{pmatrix}
a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\
a_{22} & a_{22}^{(1)} & \cdots & \cdots & a_{2n}^{(1)} \\
0 & 0 & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
0 & 0 & \cdots & a_{nn}^{(n-1)} & 0
\end{pmatrix}$$

Obtain $L$ and $U$: 

$$A^{(n-1)} = E_{n-1}A^{(n-2)} = E_{n-1}E_{n-2}A^{(n-3)} = \ldots = E_{n-1}E_{n-2}E_{n-3} \cdots E_2E_1A.$$ 

Set: 

$$U = A^{(n-1)}, \quad L = E_{n-1}E_{n-2} \cdots E_2E$$

Then from before we have $U = LA$. Because each $E_k$ is a unit lower triangular matrix lower triangular matrix having is along the diagonal), so is the matrix $L$: therefore, $L^{-1}$ exists. (Note that the product of two triangular matrices. One type is a triangular matrix of the same type).

Set $L = L^{-1}$. Then the equation $U = L^{-1}A$ becomes $A = LU$.

This factorization of $A$ is known as LU factorization.

**Definition 8.**

The entries $a_{11}, a_{22}^{(1)}, \ldots, a_{nn}^{(n-1)}$ are called pivots, and the preceding process of obtaining LU factorization is known as Gaussian elimination without row interchanges. It is commonly known as Gaussian elimination without pivoting. The process is named after the German mathematician and astronomer Karl Friedrich Gauss (1777-1855).

**Gaussian Elimination with partial pivoting**

The process consists of $n – 1$ steps.

**Step 1.**

Scan the first column of $A$ to identify the largest element in magnitude in that column. Let be $a_{11}$. 

* Form a permutation matrix $P_1$ by interchanging the rows 1 and $r_1$ of the identity matrix and leaving the other rows unchanged.

* Form $P_1A$ by interchanging the rows $r_1$ and 1 of $A$.

* Find an elementary lower triangular matrix $M_1$ such that $A^{(1)} = M_1P_1A$ has zeros below the (1,1) entry on the first column.

It is sufficient to construct $M_1$ such that $M_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} = \begin{pmatrix} * \\ \vdots \\ 0 \end{pmatrix}$. 


Note that
\[
M_1 = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
m_{21} & 1 & 0 & \ldots & 0 \\
m_{31} & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
m_{n1} & 0 & 0 & \ldots & 0 & 1
\end{pmatrix}
\]

Where \( m_{ij} = -a_{ij} / a_{ii}, m_{ii} = -a_{ii} / a_{ii} \), and \( a_{ij} \) refers to the \((i,j)\)th entry of the permuted matrix \( P_1A \). Save the multipliers \( m_{i}, i = 2, \ldots, n \) and record the row interchanges.

\[
A^{(1)} = \begin{pmatrix}
* & * & \ldots & * \\
0 & * & \ldots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & * & \ldots & * 
\end{pmatrix}
\]

**Step 2.**

Scan the second column of \( A^{(1)} \) below the first row to identify the largest element in magnitude in that column. Let the element be \( a_{2,2}^{(1)} \). From the permutation matrix \( P_2 \) by interchanging the rows 2 and \( r \) of the identity matrix and leaving the other rows unchanged. From \( P_2A^{(1)} \). Next, find an elementary lower triangular matrix \( M_2 \) such that \( A^{(2)} = M_2P_2A^{(1)} \) has zeros below the \((2,2)\) entry on the second column. \( M_2 \) is constructed as follows. First, construct an elementary matrix \( \hat{M}_2 \) of order \((n-1)\) such that

\[
\hat{M}_2 = \begin{pmatrix}
a_{22} \\
a_{32} \\
\vdots \\
a_{n2}
\end{pmatrix}
\begin{pmatrix}
(a_{22}) & 0 \\
0 & 0 \\
\vdots & \vdots \\
0 & 0
\end{pmatrix}
\]

Then define

\[
M_2 = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & m_{n2} & 1 & \ldots & 0 \\
0 & m_{n2} & 0 & \ldots & 0 
\end{pmatrix}
\]

Note that \( a_{ij} \) refers to the \((i, j)\)th entry of the current matrix \( P_2A^{(1)} \). At the end of step 2, we will have

\[
A^{(2)} = M_2P_2A^{(1)} = \begin{pmatrix}
* & * & \ldots & * \\
0 & * & \ldots & * \\
0 & 0 & * & \ldots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & * & \ldots & * 
\end{pmatrix}
\]

Where \( m = -a_{ij} / a_{jj}, i = 3, 4, \ldots, n \). Save the multipliers \( m_{i} \) and record the row interchange.
**Step k.**

In general, at the kth step, scan the entry of the kth column of the matrix \( A^{(k-1)} \) below the row \((k-1)\) to identify the pivot \( a_{k,k} \), form the permutation matrix \( P_k \), and find an elementary lower triangular matrix \( M_k \) such that \( A^{(k)} = M_k P_k A^{(k-1)} \) has zeros below the \((k,k)\) entry on the kth column. \( M_k \) is constructed first by construction \( \tilde{M}_k \) of order \((n-k+1)\) such that

\[
\tilde{M}_k = \begin{pmatrix}
a_{kk} & \cdots & a_{kn}
0 & \cdots & 0
\vdots & \ddots & \vdots
0 & \cdots & 0
\end{pmatrix}
\]

and then defining

\[
\hat{M}_k = \begin{pmatrix}
I_{k-1} & 0 \\
0 & \tilde{M}_k
\end{pmatrix}
\]

Where 0 is a matrix of zeros. The elements to the \((i, k)\)th entries of the matrix \( P_k A^{(k-1)} \).

**Step n − 1.**

At the end of the \((n-1)\)st step, the matrix \( A^{(n-1)} \) will be an upper triangular matrix.

Form \( U \):

Set

\[
A^{(n-1)} = U
\]

Then

\[
U = A^{(n-1)} = M_{n-1} P_{n-1} A^{(n-2)} = M_{n-1} P_{n-1} M_{n-2} P_{n-2} A^{(n-3)} = \cdots = M_{n-1} P_{n-1} M_{n-2} P_{n-2} \cdots M_2 P_2 M_1 P_1 A .
\]

Set:

\[
M_{n-1} P_{n-1} M_{n-2} P_{n-2} \cdots M_2 P_2 M_1 P_1 = M .
\]

Then we have, from the preceding, the following factorization of \( A \): \( U = MA \).

**Gaussian Elimination with Complete Pivoting**

In Gaussian elimination with complete pivoting, at the kth step, the search for the pivots is made among all the entries of the submatrix below the first \((k-1)\) rows. Thus, if the pivot is \( a_{rs} \), to bring this pivot to the \((k,k)\) position, the interchange of the rows \(r\) and \(k\) has to be followed by the interchange of the columns \(k\) and \(s\). This is equivalent to premultiplying the matrix \( A^{(k-1)} \) by a permutation matrix \( P_k \) obtained by interchanging rows \(k\) and \(r\) and postmultiplying \( P_k A^{(k-1)} \) by another permutation matrix \( Q_k \) obtained by interchanging the columns \(k\) and \(s\) of the identity matrix \( I \). The ordinary Gaussian elimination is then applied to the matrix \( P_k A^{(k-1)} Q_k \); that is, an elementary lower triangular matrix \( M_k \) is sought such that the matrix \( A^{(k)} = M_k P_k A^{(k-1)} Q_k \).

Has zeros on the kth column below the \((k,k)\) entry. The matrix \( M_k \) can, of course, be computed in two smaller steps as before. At the end of the \((n-1)\)st step, the matrix \( A^{(n-1)} \) is an upper triangular matrix.

Set:

\[
A^{(n-1)} = U .
\]
Stability of Gaussian Elimination

Definition 9.
The growth factor $\rho$ is the ratio of the largest element (in magnitude) of $A$, $A^{(1)}, \ldots, A^{(n-1)}$ to the largest element (in magnitude) of $A$: $\rho = \frac{\max(\alpha_i|a_{ij}|, a_{kj})}{\alpha}$

Where $\alpha = \max_{i,j}|a_{ij}|$ and $\alpha_k = \max_{i,j}|a_{ij}|$

Example 3.
Let $A = \begin{pmatrix} 0.0001 & 1 \\ 1 & 1 \end{pmatrix}$

(a) Gaussian elimination without pivoting gives
$A^{(1)} = U \approx \begin{pmatrix} 0.0001 & 1 \\ 0 & -10^4 \end{pmatrix}$; $\max|a_{ij}| = 10^4$, $\max|a_{ij}| = 1$, $\rho = \text{the growth factor} = 10^4$

(b) Gaussian elimination with pivoting yields
$A^{(1)} = U \approx \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; $\max|a_{ij}| = 1$, $\max|a_{ij}| = 1$, $\rho = \text{the growth factor} = 1$

Growth Factor of Gaussian Elimination for Complete Pivoting

For Gaussian elimination with complete pivoting,
$\rho \leq \left\{ n, 2, 3^\frac{1}{2}, 3^\frac{1}{3}, 4^\frac{1}{4}, \ldots, n^\frac{1}{n-1} \right\}^{\frac{1}{2}}$

Growth Factor of Gaussian Elimination for Partial Pivoting.

For Gaussian elimination with partial pivoting, $f \leq 2^{n-1}$, that is, $\rho$ can be as big as $2^{n-1}$

Growth Factor and Stability of Gaussian Elimination without Pivoting

$\rho$ can be arbitrarily large, except for Gaussian elimination without Pivoting, for a few special cases, as we shall see later, such as symmetric positive definite matrices. Thus, Gaussian elimination without Pivoting is, in general, a completely unstable algorithm.

Definition 10.
An $n \times n$ real matrix $A = [a_{ij}]$ is an M-matrix if $a_{ii} \geq 0$, if $a_{ij} \leq 0$ for all $i \neq j$, and if $A^{-1} \geq 0$. (i.e., $A^{-1}$ is an $n \times n$ matrix with nonnegative entries).

Example 4.
The following matrix $M$ is an M – matrix.

$$M = \begin{pmatrix}
6 & -1 & 0 & 0 & 0 & 0 \\
-1 & 6 & 0 & -1 & 0 & -1 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & -1 \\
-1 & 0 & 0 & 0 & -1 & 6
\end{pmatrix}$$
Definition 11.
The comparison matrix \( M^{(i)} = [a_{ij}] \) of an arbitrary \( n \times n \) complex matrix \( A = [a_{ij}] \) is defined by
\[
\alpha_{ij} = \begin{cases} 
    [a_{ij}] & \text{if } i = j \\
    -[a_{ij}] & \text{if } i \neq j 
\end{cases}
\]

Example 5.

Let \( A = \begin{pmatrix} 7 & 4 & 2i & 2i \\ 7 & 3 & i & 1 \\ 8 & 7 & i & 3 \\ 7 & 4 & i & 1 & 3 \end{pmatrix} \), Obviously \( M(A) = \begin{pmatrix} 7 & -4 & 2 & -2 \\ 7 & 3 & -1 & -1 \\ 8 & 7 & -1 & 3 & -1 \\ 7 & 4 & -1 & -1 & 3 \end{pmatrix} \).

Definition 12.
We define an \( n \times n \) complex matrix \( A \) to be an H-matrix iff \( M(A) \) is an M-matrix.

Definition 13.
An \( n \times n \) complex matrix \( A \) is an nonsingular H-matrix if \( M(A) \) is an nonsingular M – matrix.

Example 6.
The matrix \( A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & 0 \\ -2 & 0 & 2 \end{pmatrix} \) is an H-matrix.

With the exception of M-matrices and comparison matrices, we assume that any matrix in this paper has complex – valued entries. Let \( P \) denote any set (Possibly empty) of ordered pairs of integers \((i, j)\), with \( 1 \leq i, j \leq n \) and \( i \neq j \). It is convenient to call \( P_o \) the collection of all such sets \( P \).

Then, given any \( n \times n \) matrix \( A = [a_{ij}] \) and any \( P \in P_o \) the incomplete LU- factorization of \( A \), depending on \( P \), consists in the following process, given by Meijerink and Van Der Vorst [7].

\[
\begin{align*}
A_0 &= A \\
\bar{A}_k &= A_{k-1} \pm N_k \\
A_k &= L_k \bar{A}_k
\end{align*}
for \( k = 1, 2, \ldots, n-1 \)
\]

Where, if we set \( A_k = [a^k_{ij}] \) and \( \bar{A}_k = [\bar{a}^k_{ij}] \), the matrices \( N_k = [n^k_{ij}] \) and \( L_k = [l^k_{ij}] \) are defined by
\[
\begin{align*}
n^k_{ij} &= -a^k_{ij} & \text{if } (i, k) \in P \\
n^k_{k,j} &= -a^k_{k,j} & \text{if } (k, j) \in P \\
n^k_{i,j} &= 0 &\text{Otherwise}
\end{align*}
\]
\[
\begin{align*}
l^k_{i,k} &= -\bar{a}^k_{i,k} & \text{if } (i, k) \in P \\
l^k_{k,j} &= -\bar{a}^k_{k,j} & \text{if } (k, j) \in P \\
l^k_{i,j} &= 1 &\text{for } i = k + 1, \ldots, n \\
l^k_{i,j} &= 0 &\text{for } i = 1, 2, \ldots, n \\
l^k_{i,j} &= 0 &\text{Otherwise}
\end{align*}
\]
At the step $n-1$ we set

$$
\begin{align*}
U &= A_{n-1} \\
L &= \left( \prod_{k=1}^{n-1} L_{n-k} \right)^{-1} \\
N &= \sum_{k=1}^{n-1} N_k
\end{align*}
$$

And we obtain $A = LU - N$, where $L$ and $U$ are respectively a lower triangular $n \times n$ matrix with all diagonal entries unity and an upper triangular $n \times n$ matrix.

**Stability for M-matrices**

Meijerink and van der Vorst [6] have studied the stability of incomplete LU-factorization for M-matrices.

**Theorem 1.** [7].

If $A$ is an $n \times n$ M-matrix, then there exists for every $P \in P_n$ a lower triangular matrix $L$, with unit diagonal, an upper triangular matrix $U$ and a matrix $N$, such that $A = LU - N$. The factors $L$ and $U$ are unique.

The following example is an M-matrix which does not have an LU factorization.

$$
A = \begin{pmatrix}
6 & -1 & 0 & 0 & 0 & 0 \\
-1 & 6 & 0 & -1 & 0 & -1 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & -1 \\
-1 & 0 & 0 & 0 & -1 & 6
\end{pmatrix}
$$

**Theorem 2.** [7].

If $A$ is an M-matrix, then the construction of an incomplete LU-factorization is at least as stable as the construction of the complete factorization $A = LU$ without Pivoting.

**Example 7.**

To illustrate that a large growth factor $\rho$ is possible in H-matrices, consider the matrix

$$
M = \begin{pmatrix}
2 & 0 & -x \\
-x & x & -1 \\
0 & -1 & x
\end{pmatrix}
$$

Which is a nonsingular M-matrix for all $x > \sqrt{2}$. Gaussian elimination without pivoting applied to $M$ yields an unbounded growth factor $\rho$ of $\frac{x}{2} + \frac{1}{x}$ whereas with cdd pivoting $\rho = 1$.

**Stability for H-matrices**

First let us recall some results about the existence of the incomplete LU-factorizations of H-matrices.

**Theorem 3.**

If $A$ is an $n \times n$ H-matrix, then there exists, for every $P \in P_{n}$, a lower triangular matrix $L$, with unit diagonal, an upper triangular matrix $U$ and a matrix $N$, such that $A = LU - N$. The factors $L$ and $U$ are unique.
Theorem 4. [2].
Let \(A\) be an \(n \times n\) H-matrix, let \(M(A) = (m_{ij})\) denote its comparison matrix, and suppose that \(M(A)\) admits an LU factorization into M-matrices. Then \(A\) admits an LU factorization, and if \(A^{(k)}\) and \(M^{(k)}\), respectively, denote the reduced matrices of \(A\) and \(M(A)\), then \(M^{(k)} \leq M(A^{(k)})\), \(1 \leq k \leq n-1\).

We note that the converse of Theorem 4 is false: if \(A\) is an H-matrix which admits an LU factorization, then \(M(A)\) doesn’t necessarily admit an LU factorization, and if \(k\) denote respectively the reduced matrices of \(A\) and \(M(A)\), then \(1 \leq k \leq n-1\).

Theorem 5. [7].
Let \(A = [a_{ij}]\) be an \(n \times n\) H-matrix and \(B = [b_{ij}]\) an \(n \times n\) M-matrix (in particular \(B = M(A)\)) such that \(B \leq M(A)\). For \(1 \leq k \leq n-1\), let \(\tilde{A}_k = [\tilde{a}_{ij}]\) and \(\tilde{B} = [\tilde{b}_{ij}]\) be the matrices that arise respectively form \(A\) and \(B\) by eliminating the \(k\)th column using the \(k\)th row. Then \(\tilde{B} \leq M(\tilde{A}_k)\).

We have the following theorem for the stability of the incomplete LU-factorizations of an H-matrix and of its comparison matrix.

Theorem 6. [7].
The incomplete LU-factorization of an H-matrix \(A\) is at least as stable as the incomplete LU-factorization of its comparison matrix \(M(A)\).

Theorem 7.
The incomplete LU-factorization of an H-matrix \(A\) is at least as stable as the complete LU-factorization of its comparison matrix \(M(A)\) without pivoting.

Remark 1. [7].
We cannot say that the incomplete LU-factorization is at least as stable as the complete LU-factorization of an H-matrix

Let \(Y\) be an M-matrix, and a matrix \(X\) satisfy the condition \(M(X) \geq Y\). Then \(X\) is an H-matrix and \(|X^{-1}| \leq Y^{-1}\).

Theorem 8. [7].
Let \(A = [a_{ij}]\) be an \(n \times n\) H-matrix and \(B = [b_{ij}]\) an \(n \times n\) M-matrix (in particular \(B = M(A)\)) such that \(B \leq M(A)\). For \(1 \leq k \leq n-1\), if \(\tilde{A} = [\tilde{a}_{ij}]\) and \(\tilde{B} = [\tilde{b}_{ij}]\) denote respectively the results of the \(k\)th Gaussian elimination of \(A\) and \(B\), then we have

i) \(\tilde{B}\) is an M-matrix

ii) \(\tilde{A}\) is an H-matrix

iii) \(\tilde{B} \leq M(\tilde{A})\)

iv) \(|\tilde{A}^{-1}| \leq [M(\tilde{A})]^{-1} \leq \tilde{B}^{-1}\)
Theorem 9. [1].
Let $A$ be an $n \times n$ H-matrix. There exists a permutation matrix $p$ such that $PAP^T$ admits an $LU$ factorization using Gaussian elimination with cdd pivoting.

2 Conclusion

The purpose of this paper is to show that the incomplete LU factorizations of the H-matrices by using the incomplete LU- factorizations of an H-matrix are at least as stable as the complete LU- factorizations of its comparison matrix. We give also some new characterizations of the H-matrices in connection with their incomplete LU-factorizations.

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