Development of a Joint Economic Lot Size Model for Vendor–Buyer with Normal Demand and Empirical Delivery Time with Service level and Incremental Discount

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Abstract
This study addresses the single-product joint vendor-buyer lot-sizing problem with normal and empirical distributions for demand and delivery time respectively. A service level constraint is assumed for the buyer in which unsatisfied demand might be lost or backordered. The vendor produces products at a certain rate and as a fraction of the received order and, subsequently, ships the produced product to the buyer. The shipments are assumed to be equal in size. Also, to encourage the buyer to place further orders, the vendor adopts the incremental discount policy. The purpose is to determine the number of shipments delivered by the vendor to the buyer, as well as buyer’s reorder point, buyer’s order quantity, and buyer’s safety stock, so as to minimize the average costs incurred by both the buyer and the vendor. Once the problem has been formulated, the particle swarm optimization (PSO) algorithm was implemented to solve it.

Keywords: Vendor-buyer integration, Stochastic demand, Stochastic lead-time, Incremental discount, Particle swarm optimization.

1 Introduction

Consider a supply chain consisting of a single vendor and a single buyer. Assume that delivery times follow an empirical distribution function. As soon as the inventory position reaches the reorder point, the Q accumulation size is ordered by the buyer according to the proposed incremental discount intervals. Then, the vendor starts producing goods at the rate of P. Also, the production volumes are multiples of Q. Upon completion of each production batch; it is shipped to the buyer for a certain shipment cost. Since the demand is stochastic, the buyer might confront shortage and must therefore take into account shortage costs. Both types of shortages, including backordered and lost sales, is allowed in the model. To minimize the probability of confronting shortages, a certain safety stock must be considered. The purpose is to
determine the number of shipments delivered by the vendor to the buyer, the buyer’s reorder point, the buyer’s safety stock and the buyer’s order size with respect to the given discount intervals in order to minimize the total vendor-buyer expected cost. Investigation of the Joint Economic Lot-Size (JELS) model started by Goyal [1], and Banerjee [2] presented an important idea in this regard by introducing the lot-for-lot model. According to this model, upon completion of each production batch, the vendor would ship it to the buyer. Lu [3] considered the JELS problem case in which the vendor has a monopoly on product and attempt to minimize the buyer’s cost. Hill [4] assumed the problem that is included a vendor, a buyer, and a single product type. He proved that by assuming non-identical products, more savings were obtained in compared to the case in which the products were identical. He also proved that the shipments increased should be proportional to a multiplication factor between 1 and the number representing production rate to demand rate ratio. Ben-Daya and Hariga [5] introduced the probable normal demands into the model and assumed that there was a linear relation between delivery time and size of the accumulated products. Pan and Hsiao [6] considered the joint model in the case in which backordered and probable normal demand were allowed. Furthermore, they assumed that delivery time and shortages were negotiable. Seliaman and Ahmad [7] took into account the three-level supply chain with probable buyer’s demand. Sang and Dinwoodie [8] considered a three-level supply chain with Poison’s demand and exponential delivery time. In their study, they investigated the effect of vendor manage inventory (VMI) in the joint model, as well as the allowable backordered shortage. Glock [9] extended the model proposed by Ben-Daya and Hariga [5] to permitted batch shipments increasing by a fixed factor. Taleizadeh et al. [10] studied the joint vendor-buyer lot-size model in the multi-product case involving probable normal demand as well as both backordered and lost sale. They used meta-heuristic particle swarm optimization (PSO), genetic (GA), and simulated annealing (SA) algorithms. By considering probable demand, Glock [11] devised strategies to reduce delivery time including preparation time reduction, transfer time reduction, accumulated size reduction and production rate increase. Lee et al. [12] considered a lot-sizing problem with multiple suppliers, multiple periods and quantity discounts. They used meta-heuristic genetic algorithms (GA) to solve the model. Hoque [13] studied a joint vendor-buyer lot-size model with a normal distribution of lead-time for equal and unequal-sized batches of a lot in which backorder shortage was allowed. In this paper, a single vendor-single buyer inventory problem is considered in which the demand and lead-time follow a normal and empirical probability distribution, respectively. Furthermore, a combination of backorder and lost sale is considered. Also, to encourage the buyer to place further orders, we used incremental discount policy and a service level constraint is assumed. Moreover, the PSO algorithm is proposed to solve the model.

2 Preliminaries and notations

In this section, after introducing the parameters and variables of the model, including the total cost of buyer and vendor, the service level constraint and the total supply chain costs are given.

2.1. Decision Variables

The decision variables are as follows:

\( r \): Buyer’s reorder point

\( n \): Number of shipments from the vendor to the buyer

\( Q \): size of equal shipments from the vendor to the buyer

\( ss \): Buyer’s safety stock

2.2. Parameters

The parameters are as follows:

\( D \): Buyer’s annual average demand
\( \mu_B \): Demand mean during the lead-time of the buyer
\( \sigma_B \): Demand variance during the lead-time of the buyer
A: Ordering cost of the buyer
\( A' \): Transportation cost of each shipment to the buyer
\( A^p \): Setup cost for the vendor
\( h_b \): Unit holding cost at the buyer’s site
\( h_v \): Unit holding cost at the vendor’s site
I: Maximal inventory level of the vendor
\( \bar{B}(r) \): Expected amount of the stock outs at the buyer’s site
\( \beta \): The percentage of the demand that is backordered at the buyer
B: Expected amount of the back ordered demand
S: Expected amount of the lost demand
L: Delivery time of each shipment from vendor to buyer
\( F_L(D) \): The demand CDF during delivery time at the buyer
\( \pi \): Unit backordered cost
\( \pi' \): Unit lost demand cost
\( s_f \): The lower bound of the service level for the buyer
N: The number of breakpoints in the range proposed discounts
U: Purchase price of the product
\( C_{A_b} \): Buyer’s expected total ordering cost
\( C_{A_v} \): Vendor’s expected total set up cost
\( C_{T_b} \): Buyer’s expected total transportation cost
\( C_{H_b} \): Buyer’s expected total holding cost
\( C_{H_v} \): Vendor’s expected total holding cost
\( C_{B_b} \): Buyer’s expected total backordered cost
\( C_{S_b} \): Buyer’s expected total lost demand cost
\( C_{M_b} \): Buyer’s expected purchasing cost

3 Main section

3.1. The Buyer’s Cost

The average buyer’s cost is equal to the sum of ordering costs sustained due to transportation, storage, backordered shortage, lost sale shortage, purchase, and is calculated from Eq. (3.1):
\[
TC_{\text{buyer}} = C_{A_b} + C_{T_b} + C_{H_b} + C_{B_b} + C_{S_b} + C_{M_b} \tag{3.1}
\]

The identical size shipment policy was used in this study. Figure 1 shows an example of five identical shipments (Hill [4]), where upon issuance of order Q by the buyer, the vendor produces nQ products at a rate of P (P>D), shipping the first Q to the buyer just as its production is completed. According to Fig. 2, the time intervals between two consecutive orders and two consecutive shipments are nQ/D and Q/D respectively. As a result, the cost of the transportation and order can be computed as Eqs. (3.2) And (3.3):
Figure 1: An order with 5 identical shipments

\[ C_{Ab} = \frac{AD}{nQ} \]  \hspace{1cm} (3.2)
\[ C_{Tb} = A^T \frac{D}{Q} \]  \hspace{1cm} (3.3)

The safety stock \((ss)\) can be obtained from Eq. (3.4) considering both backordered and lost demands.

\[ ss = \beta ss^{\text{Backorder}} + (1 - \beta) ss^{\text{Lost-sale}} \]  \hspace{1cm} (3.4)

Referring Appendix A, we can compute \(ss^{\text{Lost-sale}}\) and \(ss^{\text{Backorder}}\) as in Eqs. (3.5) and (3.6):

\[ ss^{\text{Backorder}} = \sum_{VL_i} \left[ \frac{\sqrt{L_i} \sigma_D}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{\sqrt{L_i} \mu_D}{\sigma_D} \right)^2} - L_i \left( \frac{\mu_D}{2} + \frac{r}{2} \right) \right] P(L = L_i) \]  \hspace{1cm} (3.5)

\[ ss^{\text{Lost-sale}} = \sum_{VL_i} \left[ \frac{\sqrt{L_i} \sigma_D}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{\sqrt{L_i} \mu_D}{\sigma_D} \right)^2} - L_i \left( \frac{\mu_D}{2} + \frac{r}{2} \right) \right] + \frac{\sqrt{L_i} \sigma_D G_u(z_{sl})}{\sqrt{2\pi}} \]  \hspace{1cm} (3.6)

According to Eqs (3.4-3.6) and Appendix A, \(ss\) can be given by Eq. (3.7)

\[ ss = \sum_{VL_i} \left[ \frac{\sqrt{L_i} \sigma_D}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{\sqrt{L_i} \mu_D}{\sigma_D} \right)^2} - L_i \left( \frac{\mu_D}{2} \right) \right] P(L = L_i) \] + \(1 - \beta\) \\[ \sum_{VL_i} \left[ \frac{\sqrt{L_i} \sigma_D G_u(z_{sl})}{\sqrt{2\pi}} \right] P(L = L_i) \] + \(\frac{r}{2}\)  \hspace{1cm} (3.7)

Considering Eq. (3.7), we can obtain the buyer holding cost from Eq. (3.8)

\[ C_{Hb} = h_b \left( \frac{Q}{2} + ss \right) \]  \hspace{1cm} (3.8)

The stock-out costs consist of backordered and lost demand costs. When both demand and delivery times are stochastic, the expected amount of the stock out can be given by Eq. (3.9)

\[ \bar{b}(r) = \sum_{VL_i} (r^{\frac{1}{2}} (D - r) \frac{1}{\sigma_D \sqrt{2\pi}} e^{-\frac{(r - \mu_D)^2}{2\sigma_D^2}} d_D) P(L = L_i) \]  
\[ = \sum_{VL_i} (\sigma_D G_u(z_{sl})) P(L = L_i) \]  
\[ = \sum_{VL_i} \left( \sqrt{L_i} \sigma_D G_u(z_{sl}) \right) P(L = L_i) \]  \hspace{1cm} (3.9)

The value of \(C_{Bb}\) and \(C_{Sb}\) can be obtained by Eq. (3.10) and Eq. (3.11):

\[ C_{Bb} = \frac{\pi BD \bar{b}(r)}{nQ} \]  \hspace{1cm} (3.10)
\[ C_{Sb} = \frac{\pi (1 - \beta) D \bar{b}(r)}{nQ} \]  \hspace{1cm} (3.11)

To encourage the buyer to order further products, the incremental discount policy was adopted. According to this policy, the prices vary at different intervals and the items in are purchased each interval at the price
designated for that particular interval. Table 1 describes the proposed prices in the incremental discount method.

<table>
<thead>
<tr>
<th>Unit Price for Total Order Quantity</th>
<th>Order Quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_1$</td>
<td>$0 &lt; Q \leq q_1$</td>
</tr>
<tr>
<td>$u_2$</td>
<td>$q_1 &lt; Q \leq q_2$</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>$u_{n-1}$</td>
<td>$q_{n-1} &lt; Q \leq q_n$</td>
</tr>
<tr>
<td>$u_n$</td>
<td>$q_n &lt; Q$</td>
</tr>
</tbody>
</table>

Table 1: Proposed Prices for the incremental Discount Case

Unit price of product in the incremental discount case is determined for each interval from Table 2 by duly considering the concept of mathematical expectation.

<table>
<thead>
<tr>
<th>Average Unit Price of Individual Product</th>
<th>Order Quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_1$</td>
<td>$0 &lt; Q \leq q_1$</td>
</tr>
<tr>
<td>$u_1q_1 + u_2(Q - q_1) \over Q$</td>
<td>$q_1 &lt; Q \leq q_2$</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>$u_1q_1 + u_2(q_2 - q_1) + \cdots + u_n(Q - q_n) \over Q$</td>
<td>$q_n &lt; Q$</td>
</tr>
</tbody>
</table>

Table 2: Unit Price of Product on the Purchase Order at each Purchase Interval in the incremental Discount Case

The buyer’s purchase cost is calculated from Eq.(3.12) by considering the selected interval:

$$C_{M_b} = \sum_{i=1}^{N} D_u x_i$$

(3.12)

**3.2. The Vendor’s Costs:**

The vendor’s total cost consists of the preparation and holding costs. It can be given by Eq. (3.13)

$$TC_{Vendor} = C_{A_v} + C_{H_v}$$

(3.13)

According to Ban-Daya and Hariga [5] the vendor’s holding and setup cost is obtained from Eqs. (3.14) and (3.15).

$$C_{A_v} = \frac{A_{PD}}{nQ}$$

(3.14)

$$C_{H_v} = h_v \frac{Q}{2} \left[n \left(1 - \frac{D}{P}\right) - 1 + \frac{2D}{P}\right]$$

(3.15)

Therefore, the vendor’s total cost can be obtained from Eq. (3.16)

$$TC_{Vendor} = C_{A_v} + C_{H_v} = \frac{A_{PD}}{nQ} + h_v \frac{Q}{2} \left[n \left(1 - \frac{D}{P}\right) - 1 + \frac{2D}{P}\right]$$

(3.16)

**3.3. Service Level Constraint:**

Stock-out in an inventory system with continuous review policy occurs when the demand during lead -time exceeds the reorder point given by (3.17)

$$r \geq \sum_{t_{L_1}} \left(\sqrt{L_t \sigma_{D} z_{st}} + L_t \mu_D \right) p(L = L_t)$$

(3.17)

Furthermore, $ss$ should be positive ($ss \geq 0$); therefore

$$ss \geq 0$$

(3.18)
As a result, \( r \) can be computed from (3.20)

\[
\begin{align*}
\sum_{i=1}^{N} \left[ -\frac{\sqrt{\mu_D} L_i}{\sqrt{2\pi}} e^{-\left(\frac{1}{2}\left(\frac{\sqrt{\mu_D}}{\sigma_D}\right)^2 \right.} - L_i \left( \frac{\mu_D}{2} \right) \right] P(L = L_i) \\
+ (1 - \beta) \sum_{i=1}^{N} \left[ \left( \frac{L_i \sigma_D}{G_u} \right) \right] P(L = L_i) \bigg] + \frac{r}{2} \geq 0
\end{align*}
\]

(3.19)

As a result, \( r \) can be computed from (3.20)

\[
\begin{align*}
r \geq -2 \sum_{i=1}^{N} \left[ -\frac{\sqrt{\mu_D} L_i}{\sqrt{2\pi}} e^{-\left(\frac{1}{2}\left(\frac{\sqrt{\mu_D}}{\sigma_D}\right)^2 \right.} - L_i \left( \frac{\mu_D}{2} \right) \right] P(L = L_i) \\
-2(1 - \beta) \sum_{i=1}^{N} \left[ \left( \frac{L_i \sigma_D}{G_u} \right) \right] P(L = L_i) \\
\end{align*}
\]

(3.20)

### 3.4. The Mathematical Model:

According to (3.1)-(3.20), the model can be stated as (3.21)-(3.27)

\[
TC = \sum_{i=1}^{N} \frac{A_D}{n_i} x_i + \sum_{i=1}^{N} h_p \left( \frac{Q_i}{2} + SS \right) x_i + \sum_{i=1}^{N} \frac{(\pi \beta + \pi(1 - \beta)) dB(r)}{n_i} x_i + \sum_{i=1}^{N} A_D \frac{D}{Q_i} x_i \\
+ \sum_{i=1}^{N} \frac{A_P}{n_i} x_i + \sum_{i=1}^{N} h_p \frac{Q_i}{2} \left[ n \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right] x_i + \sum_{i=1}^{N} Du_i x_i
\]

(3.21)

\[
\begin{align*}
\text{St:} \\
\sum_{i=1}^{N} x_i &= 1 \\
r &\geq 2 \sum_{i=1}^{N} \left[ \left( \frac{L_i \sigma_D}{G_u} \right) \right] P(L = L_i) \\
\end{align*}
\]

(3.22)

(3.23)

\[
\begin{align*}
r &\geq -2 \sum_{i=1}^{N} \left[ -\frac{\sqrt{\mu_D} L_i}{\sqrt{2\pi}} e^{-\left(\frac{1}{2}\left(\frac{\sqrt{\mu_D}}{\sigma_D}\right)^2 \right.} - L_i \left( \frac{\mu_D}{2} \right) \right] P(L = L_i) \\
-2(1 - \beta) \sum_{i=1}^{N} \left[ \left( \frac{L_i \sigma_D}{G_u} \right) \right] P(L = L_i) \\
\end{align*}
\]

(3.24)

\[
\begin{align*}
ss &= \sum_{i=1}^{N} \left[ -\frac{\sqrt{\mu_D} L_i}{\sqrt{2\pi}} e^{-\left(\frac{1}{2}\left(\frac{\sqrt{\mu_D}}{\sigma_D}\right)^2 \right.} - L_i \left( \frac{\mu_D}{2} \right) \right] P(L = L_i) \\
+ (1 - \beta) \sum_{i=1}^{N} \left[ \left( \frac{L_i \sigma_D}{G_u} \right) \right] P(L = L_i) \bigg] + \frac{r}{2} \\
\bar{b}(r) &= \sum_{i=1}^{N} \left[ \left( \frac{L_i \sigma_D}{G_u} \right) \right] P(L = L_i) \\
Q, r, n, ss &\geq 0, \text{ Integer}
\end{align*}
\]

(3.25)

(3.26)

(3.27)

### 4. Numerical examples

The model obtained in Section 3.4 is a non-linear programming model. This is one of the most complicated problems in practical optimization (El-Sharkawi [14]). For this reason, meta-heuristic particle swarm optimization (PSO) algorithms were used for solving this model.

#### 4.1. PSO Algorithm:

The meta-heuristic particle swarm optimization (PSO) algorithm, proposed in 1995 by Kennedy and Eberhart, is an evolutionary calculation method based on the solutions population. The algorithm is a meta-heuristic method inspired by the collective behavior of natural organisms like birds and schools of fish. In these communities, there is no coordinative behavior without the central control. In this algorithm, each solution is like a bird in a group of birds called “Particle”. It is similar to the chromosome in Genetic Algorithm. All of the particles have their own specified fitness value, which is obtained from the fitness.
function. The goal, however, is to optimize the fitness value of particles. The direction of each particle is determined with respect to its velocity vector. Unlike the Genetic Algorithm, in the PSO’s Algorithm evolutionary process, there is no born particle from the old generation particle, but each particle considering its own experience and also other particles experience modifies its movement to the objective. The PSO starts with a group of random solutions (particles) and after each iteration looks for the optimum solution through updating the particles. The algorithm consists of three main steps, namely, determination of particles positions and velocities, updating particles velocities, and updating particles positions. These steps are described in the following section.

4.1.1. First Step: Determination of each Particle’s Position and Speed

A particle refers to a point in the designed space and each particle is a candidate solution to the problem. Each particle has a particular position and speed indicating its direction and speed of flight. At each iteration, each particle changes its position based on the updated velocities. Particle type is related to the number of problem variables. The variables in the present study are the following five decision variables: a discount interval \( x_i \), binary valuable (which adopts a value of either 0 or 1), buyer’s purchase quantity in the selected interval \( Q_i \) (an integer), buyer’s reorder point \( r \), buyer’s safety storage \( ss \), and the number of shipments from the vendor to the buyer \( n \). Since the decision variables are integers, the upper and the lower limits were used to assign initial values of positions and velocity vectors.

Two examples of initial particles related to the above example are produced as follows:

The first three figures show purchase quantity in the defined interval, the fourth figure denotes the reorder point quantity, and the fifth figure shows the number of shipments from the vendor to the buyer.

Moreover, two examples of these particles representing the selected interval are produced as follows:

\[ 1 \quad 0 \quad 0 \]
\[ 0 \quad 0 \quad 1 \]

Where \( i \) is the selected interval number.

4.1.2. Step 2: Updating Velocities

The velocity of each particle demonstrates its variations and is defined according to Equation (4.28):

\[ v^i(t + 1) = w v^i(t) + c_1 r_1 (x^pBest(t) - x^i(t)) + c_2 r_2 (x^gBest(t) - x^i(t)) \]  

(4.28)

In accordance with Equation (3.27), the new velocity vector for each particle is calculated based on the previous velocity of the particle \( v^i(t) \), the best position ever reached by the particle \( x^pBest(t) \), and the best position ever obtained for a neighboring particle \( x^gBest(t) \). The inertial weight factor is denoted by \( w \) (the particle tends to maintain its previous speed by this factor). Particle velocity is limited to \( V_{max} \) that controls the global searching capability of the particle swarm. In fact, the \( v^i \) elements lie in the interval \([-V_{max}, +V_{max}]\); \( r_1 \) and \( r_2 \) are random variables in the \([0, 1]\) interval, and \( c_1 \) and \( c_2 \) are learning factors. These constants show the attractiveness of each particle relative to its own position or its neighbors’ positions. Parameter \( c_1 \) is the cognitive learning factor and indicates the attractiveness of each particle relative to its own position. Parameter \( c_2 \) is the social learning factor, which indicates the attractiveness of each particle relative to its neighbors’ positions. Clerc and Kennedy method [15] was used to determine these parameters. Eqs (4.29) to (4.32) are used in this method:

\[ c_1 = \chi \cdot \phi_1 \]
\[ c_2 = \chi \cdot \phi_2 \]
\[ \phi = \phi_1 + \phi_2 \]
\[ w = \chi \]

(4.29)  
(4.30)  
(4.31)  
(4.32)
In Equation (4.30), $\phi_1$ and $\phi_2$ are determined such that $\phi > 4$. In Equation (4.32), parameter $\chi$ is calculated from Eq. (4.33):

$$\chi = \frac{2}{(\phi-2)+\sqrt{(\phi^2-4\phi)}}$$  \hspace{1cm} (4.33)

The authors proved that by using formulas (4.29) to (4.33), the algorithm exhibits a better performance. The inertial weight $w$, defined in Equation (4.32), determines what percentage of the previous velocities (from previous steps) must be maintained. A high inertial weight would make global search possible, while a low inertial weight provides the possibility for a local search. Through balanced determination of the inertia factor, better results can be obtained. In the present study, the method presented by Shi and Eberhart [16] was implemented to reduce the inertia factor at each iteration. The authors defined the inertia factor reduction as Equation (4.34):

$$w = w_{max} - \frac{w_{max} - w_{min}}{k_{max}} k$$  \hspace{1cm} (4.34)

Where $W_{min}$ and $W_{max}$ are the initial and final values of the correlation coefficient respectively, and $k_{max}$ and $k_{min}$ the maximum number of iterations and the iteration counter respectively. The values suggested for the above parameters by Shi and Eberhart [16] as well as Naka et al. [17] are $W_{max} = 0.9$ and $W_{min} = 0.4$ which have also been adapted in the present article.

4.1.3. Step 3: Updating Position

Based on its direction, each element updates its position in the decision space from Eq.(4.35):

$$x_i(t + 1) = x_i(t) + v_i(t + 1)$$  \hspace{1cm} (4.35)

And then, moves on to its new position. In Equation (4.35), $x'(t)$ is the current position, $v'(t+1)$ is the new velocity, and $v'(t+1)$ is the new position.

4.1.4. Updating the Best Particle Found

Each particle updates the solution as in Eq.(4.36):

$$f(x_i) < p_{Best}, \text{then } p^i = x^i$$  \hspace{1cm} (4.36)

Also, the best solution found in the population is updated according to Eq.(4.37):

$$f(x_i) < g_{Best}, \text{then } g^i = x^i$$  \hspace{1cm} (4.37)

Finally, the termination condition considered for this algorithm is reaching a predetermined number of iterations equal to 100.

5 Numerical examples

In Table 3, the data for the numerical example are presented. The numerical example data for solving this problem were taken from Taleizadeh et al. [11]. In this study, delivery time is also stochastic and has an empirical distribution. In Table 4, the numerical examples produced for delivery time are given. Since simultaneous delivery time and demand have not been assumed before in other studies, and since empirical distribution has not been assumed for delivery time in other studies, the authors produced the delivery time randomly. The suggested discount intervals are presented as in Table 5.
Table 4: The delivery time data

<table>
<thead>
<tr>
<th>D</th>
<th>1000</th>
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</thead>
<tbody>
<tr>
<td>μ_D</td>
<td>52</td>
</tr>
<tr>
<td>σ_D</td>
<td>6</td>
</tr>
<tr>
<td>P</td>
<td>6000</td>
</tr>
<tr>
<td>β</td>
<td>0.5</td>
</tr>
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<td>π</td>
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</tr>
<tr>
<td>H</td>
<td>12</td>
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<tr>
<td>A</td>
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</tr>
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<td>25</td>
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<td>A_p</td>
<td>400</td>
</tr>
<tr>
<td>h_b</td>
<td>6</td>
</tr>
<tr>
<td>h_v</td>
<td>5</td>
</tr>
<tr>
<td>s_l</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 3: The computational data of the example

<table>
<thead>
<tr>
<th>L_n</th>
<th>P(L_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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</tr>
<tr>
<td>2</td>
<td>0.25</td>
</tr>
<tr>
<td>3</td>
<td>0.35</td>
</tr>
<tr>
<td>4</td>
<td>0.30</td>
</tr>
</tbody>
</table>

Table 5: Suggested Discount Intervals

<table>
<thead>
<tr>
<th>Unit Price for Total Order Quantity</th>
<th>Order Quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>0 ≤ Q &lt; 500</td>
</tr>
<tr>
<td>58</td>
<td>500 ≤ Q ≤ 1000</td>
</tr>
<tr>
<td>56</td>
<td>1000 ≤ Q</td>
</tr>
</tbody>
</table>

Table 6 gives the results obtained from solving the model using the meta-heuristic PSO algorithm in two cases: joint performance and independent performance. According to the results of the joint performance case, approximate savings of 2% are obtained in the total costs of the supply chain.

Table 6: The computational results

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Q_b</td>
<td>SS</td>
<td>r</td>
<td>Cost</td>
<td>n</td>
</tr>
<tr>
<td>Independent</td>
<td>199</td>
<td>3</td>
<td>153</td>
<td>61211.47</td>
<td>-</td>
</tr>
<tr>
<td>Integrated</td>
<td>158</td>
<td>3</td>
<td>149</td>
<td>60848.54</td>
<td>3</td>
</tr>
</tbody>
</table>

Also, a sensitivity analysis was conducted on the problem parameters, i.e., transfer costs, level of service, and delivery time, to investigate their effects on the solution. Two levels were considered for each case. The levels for delivery time and the obtained results are presented in Table 7 and Table 8 respectively.

Table 7: The delivery time’s levels

<table>
<thead>
<tr>
<th></th>
<th>L_n</th>
<th>P(L_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>0.20</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>0.40</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>0.3</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.30</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>0.20</td>
</tr>
</tbody>
</table>

Table 8: The sensitivity analysis results

<table>
<thead>
<tr>
<th></th>
<th>Q</th>
<th>r</th>
<th>SS</th>
<th>n</th>
<th>z-Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_t</td>
<td>35</td>
<td>162</td>
<td>150</td>
<td>3</td>
<td>62479.01</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>156</td>
<td>149</td>
<td>3</td>
<td>62384.76</td>
</tr>
<tr>
<td>s_l</td>
<td>0.7</td>
<td>500</td>
<td>420</td>
<td>7</td>
<td>63326</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>425</td>
<td>425</td>
<td>9</td>
<td>62993.85</td>
</tr>
<tr>
<td>L</td>
<td>1</td>
<td>500</td>
<td>411</td>
<td>4</td>
<td>63341</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>500</td>
<td>284</td>
<td>4</td>
<td>63330</td>
</tr>
</tbody>
</table>
Considering Table 5, we find that the total cost of the supply chain reacts to the delivery time variations, and that the total cost increases with increased delivery time. Moreover, values obtained for reorder point and order quantity increase upon increased delivery time and level of service.

5 Conclusion

This study investigated the joint economic lot size model for vendor and buyer with probable normal demand and empirical delivery time with shipments that are identical in size. Also, constraints regarding level of service as well as lost sales and backordered have been considered in the model. In addition, the policy of incremental discount was introduced in the models as well. The purpose was to determine the number of shipments delivered by the vendor to the buyer, buyer’s reorder point, and buyer’s order size and safety inventory so that the average vendor-buyer costs can be minimized. The meta-heuristic particle swarm optimization (PSO) algorithm was used to solve the problem. Finally we proved that the integrated form has reduced the costs about 2%. The model extension for multiple buyers and vendors, considering other distribution for delivery time can be good topic for the future research.

Appendix A. Calculating Safety Stock

\[ ss = \beta s_{\text{Backorder}} + (1 - \beta)s_{\text{Lost-sale}} \]

Delivery times have an empirical PDF that is shown in table A.1

<table>
<thead>
<tr>
<th>( \mathcal{L} )</th>
<th>( L_1 )</th>
<th>( L_2 )</th>
<th>...</th>
<th>( L_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>probability</td>
<td>( P_1 )</td>
<td>( P_2 )</td>
<td>...</td>
<td>( P_n )</td>
</tr>
</tbody>
</table>

\( s_{\text{Backorder}} \): According to Figure A.1, in backorder form, the inventory level while the order is received is:

\[ E(I) = \int_0^{\infty} (r - D_t)f_t(D)d_D \]  \hspace{1cm} (A-1)
Because the delivery time is also stochastic with empirical PDF, so:

\[ s_{\text{Backorder}} = \sum v_{L_i} \left( \int_0^\infty (r - D_L) f_L(D) \, dD \right) \cdot P(L = L_i) \]
\[ = \sum v_{L_i} \left( \int_0^\infty r \cdot f_L(D) \, dD + \int_0^\infty -D_L f_L(D) \, dD \right) \cdot P(L = L_i) \]
\[ = \sum v_{L_i} \left( \frac{r}{2} - \int_0^\infty D_L \cdot \frac{1}{\sigma_{DL} \sqrt{2\pi}} \cdot e^{-\frac{(D_L - \mu_{DL})^2}{2\sigma_{DL}^2}} \, dD \right) \cdot P(L = L_i) \quad (A-2) \]

After simplifying the integral, we have:

\[ = \sum v_{L_i} \left( \frac{r}{2} - \frac{\sigma_{DL}^2}{\sqrt{2\pi}} \cdot e^{-\frac{(L_i - \mu_{DL})^2}{2\sigma_{DL}^2}} \right) \cdot P(L = L_i) \quad (A-3) \]

Moreover, the mean and variance of the lead-time is calculated as Eqs. (A-4) and (A-5)

\[ \mu_{DL} = E(D_L) = \mu_D \cdot \mu_L = L. \mu_D \quad (A-4) \]
\[ \sigma_{DL} = \sqrt{\mu_L \cdot \sigma_D^2 + \mu_D^2 \cdot \sigma_L^2} \rightarrow \sigma_{DL} = \sqrt{L \sigma_D^2 + \mu_D^2} \rightarrow \sigma_{DL} = \sqrt{L \sigma_D} \quad (A-5) \]

As a result, \( s_{\text{Backorder}} \) can be computed from (A-5)

\[ s_{\text{Backorder}} = \sum v_{L_i} \left( -\frac{\sqrt{L_i \sigma_D}}{\sqrt{2\pi}} \cdot e^{-\frac{(L_i - \mu_D)^2}{2\sigma_D^2}} - \left( \frac{L_i}{2} \right) \right) \cdot P(L = L_i) \quad (A-6) \]

\( S_{\text{lost-sale}} \). As in the lost-sale form, there is no stock-out; the net inventory (via Figure 2) is equal to (A-7)

\[ S_{\text{lost-sale}} = E(L) = \int_0^\infty (r - D_L) f_L(D) \, dD \quad (A-7) \]

![Figure A.1: The inventory level mean per unit of time with presence of lost-sale stock-out](image)

Therefore we have:

\[ s_{\text{Lost-Sale}} = \sum v_{L_i} \left( \int_0^\infty (r - D) f_L(D) \, dD \right) \cdot P(L = L_i) \]
\[ = \sum v_{L_i} \left( \int_0^\infty (r - D) f_L(D) \, dD - \int_r^\infty -(D - r) f_L(D) \, dD \right) \cdot P(L = L_i) \]
\[ = \sum v_{L_i} \left( \int_0^\infty (r - D) f_L(D) \, dD + \int_r^\infty -f_L(D) \, dD \right) \cdot P(L = L_i) \quad (A-8) \]

In the above integral, \( \int_r^\infty (D - r) f_L(D) \, dD \) is equal to \( \tilde{b} (r) \).

\[ s_{\text{Lost-Sale}} = \sum v_{L_i} \left( s_{\text{Backorder}} + \tilde{b} (r) \right) \cdot P(L = L_i) \quad (A-9) \]

Safety stock in lost-sale form is:

\[ s_{\text{Lost-Sale}} = \sum v_{L_i} \left( \frac{\sqrt{L_i \sigma_D}}{\sqrt{2\pi}} \cdot e^{-\frac{(L_i - \mu_D)^2}{2\sigma_D^2}} - L_i \left( \frac{\mu_D}{2} + \frac{r}{2} \right) + \left( \sqrt{L_i \sigma_D} \cdot G_u (z_{sl}) \right) \right) \cdot P(L = L_i) \quad (A-10) \]

With calculating \( S_{\text{lost-sale}} \) and \( s_{\text{Backorder}} \), we can define the total safety stock as follow:
\[ ss = \beta_{SS}^{Backorder} + (1 - \beta)_{SS}^{Lost\text{-}sale} \]
\[ ss = \beta \sum_{\nu\ell_i} \left[ -\frac{\sqrt{\lambda_d \sigma_d}}{\sqrt{2\pi}} e^{-\left(\frac{1}{2}\left(\frac{\sqrt{\nu\ell_i} \mu_d}{\sigma_d}\right)^2\right)} - L_i \left(\frac{\mu_d}{2}\right) + \frac{r}{2}\right] \cdot P(L = L_i) \]
\[ + (1 - \beta) \sum_{\nu\ell_i} \left[ -\frac{\sqrt{\lambda_d \sigma_d}}{\sqrt{2\pi}} e^{-\left(\frac{1}{2}\left(\frac{\sqrt{\nu\ell_i} \mu_d}{\sigma_d}\right)^2\right)} - L_i \left(\frac{\mu_d}{2}\right) + \frac{r}{2} + \left(\sqrt{L_i \sigma_d, G_u(z_{sl})}\right) \right] \cdot P(L = L_i) \] (A-11)

With simplifying the above equation, we have:
\[ ss = \sum_{\nu\ell_i} \left[ -\frac{\sqrt{\lambda_d \sigma_d}}{\sqrt{2\pi}} e^{-\left(\frac{1}{2}\left(\frac{\sqrt{\nu\ell_i} \mu_d}{\sigma_d}\right)^2\right)} - L_i \left(\frac{\mu_d}{2}\right) + \frac{r}{2}\right] \cdot P(L = L_i) \]
\[ + (1 - \beta) \sum_{\nu\ell_i} \left(\sqrt{L_i \sigma_d, G_u(z_{sl})}, P(L = L_i)\right) + \frac{r}{2} \] (A-12)

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