Abstract:
We apply the trigonometrically fitted explicit hybrid three stages method for second-order initial value problems with oscillating solutions. Through several examples the accuracy of results and the stability of the method are computed. We compare our results with the classical hybrid method and the trigonometrically fitted explicit Runge Kutta method. Our results show that trigonometrically fitted explicit hybrid method is more efficient than the classical hybrid method.

Keywords: Trigonometrically fitted; Hybrid method; Stability

1. Introduction
In this paper we are concerned about the numerical integrations the initial value problem

\[ y'' = f(x, y), \quad y(x_0) = y_0, y'(x_0) = y'_0, x \in [t_0, t_{end}] \]  

(1,1)

Where the solutions have an oscillatory behavior. This kind of initial value problem appear in many practical life problems and are of fundamentally important in both theory and applications. Therefore, it is important to develop a numerical method to approximate the solutions.

The basic idea on most numerical approaches is to develop an iterative procedure approximating the solutions. Since the cost of iteration caused by the accumulative round of errors of computer makes it fundamentally important to choose an smart
choice in the number of iterations. Mostly this leads to individual find a proper step size for each method and example. This is to address the accuracy of the results. The second main concerns of any practical problem is the stability of the numerical methods. We are assuming that a practical problem governed by Equation (1.1) is stable in the sense that any small perturbation in the initial data does not give rise a huge change in the solutions. Then, we are concerned about the stability of a numerical method, that is, how a small perturbation may influence the solutions computed by the procedure.

The hybrid method has a very long history and there are many results in the research literature, see e.g. by Gautschi (see Ref. [7]) and Lyche (see Ref. [9])

One may use many different family of functions as the fitting function in the hybrid method. The family of exponential functions is an important example, since may practical problems have such kind of behaviors. Indeed, Vanden Berghe et al. [13,14] introduced an exponentially fitted and used the explicit Runge Kutta method to solve the problem. Indeed, they reduce the second order differential equation into a first-order system. In this paper we are concerned about systems with an oscillatory nature. Franco continued the work in using the exponentially fitted function through Runge-Kutta-Nystrom method as well as pairs of embedded Runge-Kutta-Nystrom method for order 4 and 3, see [4]. Coleman [2] applied B-series theory in the hybrid method.

The oscillatory behavior of the solutions hints in using trigonometric function as the family of the fitting functions. In this paper, we basically follow [15]

The stability analysis follows the definitions given by Coleman and Ixaru [3]. The stability regions of our procedure are also depicted.

The rest of this paper is organized as follows. Section 2 derives the necessary formulas in which describes the trigonometric fitted Hybrid method

The stability of our method is presented in Section 3. The method is applied on three main examples in Section 4 illustrating the applicability of the method and the accuracy of the numerical computations is compared. Conclusion is drawn in Section 5.

2 Derivation of trigonometric fitted Hybrid method

In this section, we will extend the ideas proposed in Refs. [14,13] to the two-step hybrid methods. The two-step hybrid methods considered here have the form
\[ Y_i = (1+c_i)y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{4} a_{ij} f \left( x_n + c_j h, Y_j \right) \quad i = 1, \ldots, 4 \quad (2.1) \]

\[ y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{i=1}^{4} b_i f \left( x_n + c_i h, Y_i \right) \]

which can be represented by the Butcher tableau,

\[
\begin{array}{cccc}
   c_1 & a_{11} & a_{12} & a_{13} & a_{14} \\
   c_2 & a_{21} & a_{22} & a_{23} & a_{24} \\
   c_3 & a_{31} & a_{32} & a_{33} & a_{34} \\
   c_4 & a_{41} & a_{42} & a_{43} & a_{44} \\
   b_1 & b_2 & b_3 & b_4
\end{array}
\]

The order conditions for this kind of two-step hybrid methods can be found in Ref.[2]. Basing on the conditions, Franco derived a class of explicit hybrid methods up to order five and six with only three and four stages per step, respectively (see Ref. [6]). We consider in the following the two-step hybrid fifth-order method which is dispersive of order eight and dissipative of order five with only three function evaluations per step proposed in Ref. [6], also it can be expressed in the following Butcher tableau:

\[
\begin{array}{cccccc}
   -1 & & & & & \\
   0 & & 63 & 126651 & 900249 & - \\
   100 & 200000 & 200000 & - & - \\
   -23 & -43347640 & -4864523 & 213026000 & - \\
   25 & 916464729 & 50602347 & 8248182561 & - \\
   31 & 1675 & 1000000 & 1874161 & - \\
   13692 & 2898 & 47555739 & 8947092 & - \\
\end{array}
\]

The interval of absolute stability of the method is (0, 2.84). Using the ideas in Refs.[14,13], we require the internal stages (2) and the update stage (3) integrate exactly the linear combination of the functions \{\sin( wt), \cos( wt)\} for \( w \in R \): This leads to the following equations
Solving the above equations with the choice of \( c_3 = \frac{63}{100} \), \( c_4 = -\frac{23}{37} \) we obtain

\[
\begin{align*}
a_{3,1} &= \frac{1}{100} \frac{(-63\sin(H) + 100\sin(\frac{63}{100}H))}{H^2 \sin(H)} \\
a_{3,2} &= \frac{1}{100} \frac{(163\sin(H) - 100\sin(\frac{163}{100}H))}{H^2 \sin(H)} \\
a_{4,1} &= \frac{-43347640}{916464729} + \frac{187710632711}{33909194973000} H^2 + \cdots \\
a_{4,2} &= \frac{-4864523}{50602347} + \frac{1252608037}{15180704100} H^2 + \cdots \\
a_{4,3} &= \frac{213026000}{824812561} - \frac{1428445843}{915548264271} H^2 + \cdots \\
b_1 &= \frac{31}{13692} - \frac{126994497775441}{8665399614312000000} H^4 + \cdots \\
b_2 &= \frac{-2898}{1675} - \frac{446854874706841}{5502262945284000000} H^4 + \cdots \\
b_3 &= \frac{47555739}{10000000} + \frac{486755456725}{45145648815613731} H^4 + \cdots \\
b_4 &= \frac{1874161}{8947092} + \frac{22494654966241}{4136186949048000000} H^4 + \cdots \\
\end{align*}
\]

We remark that for \( H \to 0 \) the well-known original fifth-order method is recovered.
3 Stability analysis

In this section we discuss the stability analysis and follow definitions made by Lambert and Watson [?]. Since we are dealing with trigonometric functions with the frequency, the test equation is

\[ y''(x) = -\mu^2 y(x) \]  

(3.1)

3.1 Basic definition and property

Applying a classical s-stage explicit two-step hybrid method (2),(3) with step size h to the test equation (3.1), we obtain the following recursion relation

\[ y_{n+1} - S(V^2)y_n + P(V^2)y_{n-1} = 0, \quad V = \mu h \]  

(3.2)

with

\[ S(V^2) = 2 - V^2 b^T(I + V^2 A)^{-1}(e + c), \quad P(V^2) = 1 - V^2 b^T(I + V^2 A)^{-1}c \]

Also they can be written by

\[ S(V^2) = 2 - V^2 b^T(e + c) + V^4 b^T A(e + c) - V^6 b^T A^2(e + c) + \cdots + (-1)^{s-2} V^{2s-2} b^T A^{s-2}(e + c) \]

\[ P(V^2) = 1 - V^2 b^T c + V^4 b^T A c - V^6 b^T A^2 c + \cdots + (-1)^{s-2} V^{2s-2} b^T A^{s-2} c \]

where \( e \) is an \( s \times 1 \) vector of units, and the vectors \( c; b \) and matrix \( A \) are defined by Butcher-tableau.

**Definition 1.** [12] The quantities

\[ \Phi(V) = V - \arccos\left(\frac{S(V^2)}{2\sqrt{P(V^2)}}\right), \quad d(V) = 1 - \sqrt{P(V^2)} \]

are called the phase-lag and the dissipation, respectively. So the method is said to be dispersive of order \( q \) and dissipative of order \( p \) if

\[ \Phi(V) = O(V^{q+1}), \quad d(V) = O(V^{p+1}) \]

If \( d(V) = 0 \); we call this method is zero-dissipative. Moreover, if \( \Phi(V) = 0 \) we call this method is phase-fitted.

**Definition 2.**The polynomial

\[ \xi^2 - S(V^2)\xi + P(V^2) \]  

(3.3)

is called the stability polynomial of the difference equation (3.2). A two-step method has a periodicity interval \( I_p = (0, V_0^2) \) if the roots of its stability polynomial, \( \xi_{1,2}(V) \) are conjugate complex and \( |\xi_{1,2}(V)| = 1, \forall V \in (0, V_0^2) \) A method is called P-stable if the periodicity interval of the method is \( (0, \infty) \) (see Ref. [8]). From Definition 3.2, the
existence of a periodicity interval \( I_p = (0, V_0^2) \) is equivalent to the fact that the coefficients of the stability polynomial satisfy the conditions given by
\[
P(V^2) \equiv 1, |S(V^2)| < 2, \forall V \in (0, V_0^2).
\]
Therefore, the P-stable conditions are given by
\[
P(V^2) \equiv 1, |S(V^2)| < 2, \forall V > 0.
\]
It is noticed that \( P(V^2) < 1 \) is a necessary condition for the existence of a non-empty periodicity interval. When the method possesses a dissipation, the numerical solution remains bounded provided the coefficients of (3.3) satisfy the following conditions
\[
P(V^2) \equiv 1, |S(V^2)| < P(V^2) + 1, \forall V \in (0, V_i^2)
\]
and the interval \( I_s = (0, V_i^2) \) is called the interval of absolute stability (see Ref. [1]).

### 3.2 Stability and phase-lag analysis of the new method

The theory of Lambert and Watson was re-considered by Coleman and Ixaru (see Ref. [3]), we follow the main ideas of that paper. Applying our new method to the test equation (3.1) yields
\[
y_{n+1} - S(V^2)y_n + P(V^2)y_{n-1} = 0
\]
where
\[
S(V^2) = 2V(H)^{2}b(H)^{2}(\text{e} + c(H)) + V(H)^{2}b(H)^{2}A^{2}(\text{e} + c(H)) + \ldots + (-1)^{k}V(H)^{2}b(H)^{2}A^{2}(\text{e} + c(H))
\]
\[
P(V^2) = 1V(H)^{2}b(H)^{2}c(H) + V(H)^{2}b(H)^{2}c(H) + \ldots + (-1)^{k}V(H)^{2}b(H)^{2}c(H)
\]
Where \( e \) is a \( 4 \times 1 \) vector of units, and the vectors \( c(H); b(H) \) and matrix \( A(H) \) are given in our new method. The stability properties are determined by \( S(V^2, H) \) and \( P(V^2, H) \), and in this case the interval of absolute stability becomes a two-dimensional region. So we have the following definition.

**Definition 3.3.** A region - in the \( V \cdot H \) plane \( (V,H) > 0 \) is said to be the region of Absolute stability of the Numerov type method with the \( S(V^2, H) \) and \( P(V^2, H) \), if for all \( (V, H) \in \Omega \)
\[
P(V^2) < 1, |S(V^2)| < P(V^2) + 1
\]
Hold and, any closed curve defined by
\[
P(V^2) = 1 or |S(V^2)| = P(V^2) + 1
\]
is a stability boundary of the method. In Fig. 1 we plot the region of the absolute stability of the new method (the similar discussion can be found in Refs. [3,11]). It is
clear that the diagonal line $H = V$ (i.e. the fitted frequency $\omega$ equals the test frequency $\lambda$) is a stability boundary.

**Definition 3.4.** For the Numerov-type method with the $S(V^2, H)$ and $P(V^2, H)$, the quantities

$$\phi(V, H) = V - \arccos\left(\frac{S(V^2, H)}{2\sqrt{P(V^2, H)}}\right)$$

$$d(V, H) = 1 - \sqrt{P(V^2, H)}$$

are called the phase-lag and the dissipation, respectively. The method is said to be of phase-lag order $q$ and dissipation order $p$ if

$$\phi(V, H) = O(V^{q+1}), d(V, H) = O(V^{p+1}). \quad (3.6)$$

However, in order to analyze the properties of the phase-lag and the dissipation, the pair $V, H$ should be replaced by the new pair $V, r = H/V$. In this situation, the phase-lag and the dissipation of the new method can be easily given by

$$\Phi(V, H) = \frac{-703245532810195678}{71953665105687890625} V^3 \quad (3.7)$$

$$+ \frac{1437171808063652524r^2}{599613875880732421875}$$

$$- \frac{26873345342171456833174318456858534691}{10354659844282974328691422949523925781250} V^5$$
Thus the phase-lag order is two and the dissipation order is one for the new method.

We observe that when \( r \to 0 \), then \( H \to 0 \), the new method reduces to the classical Numerov-type method with the phase-lag order eight and the dissipation order five (see Ref. [6]). In this case, expressions (3.7) and (3.8) become

\[
\Phi(V) = -\frac{703245532810195678}{71953665105687890625} V^3 + O(V^5)
\]

\[
d(V) = -\frac{115399412}{8482550625} V^2 + O(V^4)
\]

respectively.

Thus the phase-lag order is two and the dissipation order is one, moreover, the phase-lag and dissipation constants \(-\frac{703245532810195678}{71953665105687890625}\) and \(-\frac{115399412}{8482550625}\) are the same as the corresponding constants of the classical one (see Ref. [6]). It should be noted that when the exact value of the frequency of the problem is known (i.e. the fitted frequency \( \omega \) equals the test frequency \( \lambda \)), then \( r = 1 \) and we have that \( S(V^2, V) = 2 \cos(V) \) and \( P(V^2, V) = 1 \). Consequently, when the main frequency is known the interval of periodicity is \((0, \infty)\) except for \( V = 2\pi, 4\pi, \ldots \) So the new method is almost P-stable. Moreover, the phase-lag and the dissipation are both equal to zero.

The method is then phase-fitted and zero-dissipative by Definition 3.1. In many applications the main frequencyis not exactly known. However, if a good estimate frequency is available, then we have \( r \approx 1 \). Thus, in this case, the magnitude of the phase-lag (3.7) and the dissipation (3.8) are then much smaller than those of the corresponding classical method.

**Numerical experiments:**

In this section we implement the procedure on three examples, where the exact solution is known. We, then, compare our results with [14].

**Example 1.** Let \( y'' = -\pi \sin(\pi x) \), for \( 0 \leq x \leq 10 \),where \( y(0) = 0, y'(0) = 1 \).

The exact solution of this initial value problem is \( y = \frac{\sin(\pi x)}{\pi} \).
**Example 2.** Let \( y'' = -y + 0.001\cos(x) \), for \( 0 \leq x \leq 1000 \) where \( y(0) = 1, y'(0) = 0 \). The exact solution of this initial value problem is \( u(x) = \cos(x) + 0.0005x\sin(x) \).

**Example 3.** Let \( y'' + y^3 + y = B\cos(\Omega x) \) for \( 0 \leq x \leq 300 \) where \( y(0) = 0.20042672806900; \), \( y'(0) = 0 \) with \( B = 0.002; \Omega = 1.01 \)

Its exact solution is

\[
y(x) = 0.200179477536 \cos(\Omega x) + 0.246946143 \times 10^{-3} \cos(3\Omega x) \\
+ 0.304016 \times 10^{-6} \cos(5\Omega x) + 0.374 \times 10^{-9} \cos(7\Omega x).
\]

Then we tabulated the global error evaluated at the end point of the interval, the following are the notations used in the tables:

- HMB Exponentially fitted hybrid method
- HMC classical hybrid method
- EFRK Exponentially fitted RK method using Vanden Berghe’s technique[13,14]
- \( h \) Step size

**Table 1: Comparison of the Euclidean norms of the end-point global errors**

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<tr>
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<th>HMB</th>
<th>HMC</th>
<th>EFRK</th>
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<tbody>
<tr>
<td>Problem 1</td>
<td>1</td>
<td>( \pi h )</td>
<td>( 6.5 \times 10^{-6} )</td>
<td>( 2.17 \times 10^{-2} )</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>( \pi h )</td>
<td>( 2.5 \times 10^{-14} )</td>
<td>( 1.75 \times 10^{-2} )</td>
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<tr>
<td></td>
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<td>( \pi h )</td>
<td>( 4 \times 10^{-15} )</td>
<td>( 3.3 \times 10^{-5} )</td>
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<tr>
<td></td>
<td>0.125</td>
<td>( \pi h )</td>
<td>( 6.4 \times 10^{-15} )</td>
<td>( 5.39 \times 10^{-7} )</td>
</tr>
<tr>
<td></td>
<td>0.0625</td>
<td>( \pi h )</td>
<td>( 9 \times 10^{-15} )</td>
<td>( 8.51 \times 10^{-9} )</td>
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<tbody>
<tr>
<td>Problem 2</td>
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<td>( h )</td>
<td>( 2.2 \times 10^{-4} )</td>
<td>( 1.745 \times 10^{-2} )</td>
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<tr>
<td></td>
<td>0.5</td>
<td>( h )</td>
<td>( 4.8 \times 10^{-6} )</td>
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<td></td>
<td>0.25</td>
<td>( h )</td>
<td>( 1.2 \times 10^{-7} )</td>
<td>( 1.71 \times 10^{-4} )</td>
</tr>
<tr>
<td></td>
<td>0.125</td>
<td>( h )</td>
<td>( 3.4 \times 10^{-9} )</td>
<td>( 4.88 \times 10^{-6} )</td>
</tr>
<tr>
<td></td>
<td>0.0625</td>
<td>( h )</td>
<td>( 9 \times 10^{-11} )</td>
<td>( 1.4 \times 10^{-7} )</td>
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<tbody>
<tr>
<td>Problem 3</td>
<td>1</td>
<td>( 1.01h )</td>
<td>( 7.2 \times 10^{-5} )</td>
<td>( 1.57 \times 10^{-2} )</td>
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<td></td>
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<td>( 1.01h )</td>
<td>( 1.4 \times 10^{-6} )</td>
<td>( 4.46 \times 10^{-4} )</td>
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<td>( 5.7 \times 10^{-12} )</td>
<td>( 1.31 \times 10^{-8} )</td>
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5 Conclusion

In this paper the hybrid two step method is described through trigonometric fitting function using three stages. The stability analysis and the stability regions are discussed. The method is applied on three main examples and the results are compared with the classical hybrid method and the trigonometric fitted hybrid method using two stages. In all three examples the three stages method considered in this paper works better than both the classical hybrid method and the two stages method.

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