Abstract:
In this paper a mathematical model of tumor cell population kinetics is studied. The model consists of five partial differential equations that describe the population kinetics of human tumor cell in vitro and the responses to radiotherapie or chemotherapy. This Model is solved numerically with two methods: pseudo-spectral method and finite difference method. The results are compared with expreimental Data.

Keywords: Spectral Method, Finite Difference Method ,Cell Cycle Dynamic

1 Introduction
The study of various deseases mathematicaly is progressed recently. In many cases the mathematical models are ODS or PDE. In particular study of human tumor cell in vitro and their responses to radiotherapies or chemotherapy have been subject of many papers in last decades. The numerical simulations of one of these models is performed in this paper based on the works of Jackiewicz et al.[3], and Britta Basse et al.[1,2]. Therefore two numerical methods: finite difference method(FDM) and pseudo-spectral methods are used to solve mentioned problem with various parameters. The numerical results are compared with experimental results. The paper is organized as follows. Section 2 specifed to describe the mathematical model equations. In section 3 the results of FDM and pseudo-spectral method are presented. Finally, section 4 ends the paper with discussion and conclusion.
into four distinct phases, namely, the $G_1$-phase, DNA synthesis or S-phase, $G_2$-phase, and mitosis or M-phase. The mathematical model that is used for numerical resolution was studied in [1,2]. This model describes these phases and transition rates between them and is expressed by the following linear systems of partial differential equations

$$\frac{\partial S}{\partial t}(x,t) = \epsilon \frac{\partial^2}{\partial x^2} (x,t) - \mu_S S(x,t) - g \frac{\partial S}{\partial x}(x,t) + k_i G_i(x,t) - I(x,t,T_S),$$

$$\frac{\partial G_i}{\partial t}(x,t) = 4bM(2x,t) - (k_i + \mu_{G_i}) G_i(x,t),$$

$$\frac{\partial G_2}{\partial t}(x,t) = I(x,t,T_S) - (k_2 + \mu_{G_2}) G_2(x,t),$$

$$\frac{\partial M}{\partial t}(x,t) = k_G G_2(x,t) - bM(x,t) - \mu_M M(x,t). \tag{2.1}$$

where $t \geq 0$ is time and $x$ is dimensionless relative DNA content. Dependent variables $G_1, S, G_2$ and $M$, represent the density of cells in corresponding phases. The delay term $I(x,t,T_S)$ represents cells that have been in S-phases for $T_S$ hours and are ready to be transferred to $G_2$ phases. $I(x,t,T_S)$ is the solution of following diffusion problem for $T_S = T_S$,

$$\frac{\partial I}{\partial T_S} + g \frac{\partial I}{\partial x} - \epsilon \frac{\partial^2 I}{\partial x^2} + \mu_S I = 0, \quad 0 < x < \infty, \quad t > T_S > 0,$$

$$I(x,t,0) = k_G G_1(x,t,0). \tag{2.2}$$

By using the laplace transform and Green function the analytical solution of (2.2) is obtained (See [2]) as,

$$I(x,t,T_S) = \int_0^\infty k_G G_1(\gamma, t - T_S) \gamma(T_S, x, y) dy, \quad t \geq T_S$$

$$I(x,t,T_S) = 0, \quad t < T_S.$$

where $\gamma$ is the Green function given by

$$\gamma(\tau, x, y) = \frac{\exp(-\mu_S \tau)}{2\sqrt{\pi \epsilon \tau}} \left( \exp\left(-\frac{(x - \gamma(\tau, x, y))^2}{2\epsilon \tau}\right) - (1 + \nu(\tau, x, y)) \exp\left(-\frac{(x + \gamma(\tau, x, y))^2}{4\epsilon \tau}\right) \right),$$

$$\nu(\tau, x, y) = \frac{x + y}{g \tau} \left( 1 + O(\tau^{-1}) \right).$$

The system (2.1) is subjected to following initial conditions:

$$G_i(x,0) = \frac{a_0}{\sqrt{2\pi \theta_0^2}} \exp\left(-\frac{(x - 1)^2}{2\theta_0^2}\right), \quad 0 < x < \infty,$$

$$S(x,0) = 0, \quad G_2(x,0) = 0, \quad M(x,0) = 0, \quad 0 < x < \infty,$$

$$\epsilon \frac{\partial S}{\partial x}(0,t) - gS(0,t) = \alpha, \quad t > 0.$$

where $\alpha, \beta$ and $\theta_0$ are given constants. If $\alpha = 0$, one has the zero flux condition. For solving system (2.1) two distinct sets of equations are recognized. The PDEs that can be
solved as ODEs, and the PDEs that must be solved by FDM, or spectral methods. Then, the following system of equations can be considered as an ODE system

\[
\frac{\partial G_1}{\partial t} (x,t) = 4bM(2x,t) - (k_1 + \mu_{G_1})G_1(x,t),
\]

\[
\frac{\partial G_2}{\partial t} (x,t) = I(x,t,T_s) - (k_2 + \mu_{G_2})G_2(x,t),
\]

\[
\frac{\partial M}{\partial t} (x,t) = k_2G_2(x,t) - bM(x,t) - \mu_M M(x,t).
\]  

(2.3)

By considering a partition of \([0,L]\), \(L > 2\) as \(0 = x_0 < x_1 < ... < x_N = L\) and \(t_j = j\Delta t\), \(j = 0,1,2,...,M\), and denoting:

\[
\tilde{G}_{1,j} = G_1(x,t_j), \quad \tilde{G}_{2,j} = G_2(x,t_j), \quad \tilde{M}_j = M(x,t_j)
\]

and applying, implicit Euler method for (2.3) subject to corresponding initial conditions, one has

\[
G_{2,j+1} = (1 + \Delta t (k_2 + \mu_{G_2}))^{-1} (G_{2,n} + \Delta t M_{j+1}),
\]

\[
M_{j+1} = (1 + \Delta t (b + \mu_M))^{-1} (M_j + k_2 \Delta t G_{2,j+1}),
\]

\[
G_{2,j+1} = (1 + \Delta t (k_1 + \mu_{G_1}))^{-1} (G_{1,j} + 4b \Delta t M_{j+1}^{(2)}),
\]

where \(M_j^{(2)} = M(2x,t_{j+1})\), for more details see [3]. To solve the first equation of (2-1) i.e.

\[
\frac{\partial S}{\partial t} (x,t) = \varepsilon \frac{\partial^2 S}{\partial x^2} (x,t) - \mu_s S(x,t) - g \frac{\partial S}{\partial x} (x,t) + k_1 G_1(x,t) - I(x,t,T_s),
\]

two numerical methods are used: FDM and pseudo spectral method, with Chebyshev points. The formulation of these methods is the same as [3], with a small difference that instead of Gauss-Lobato-Chebyshev points, the Chebyshev points are selected for spectral methods. Then, the implicit Euler method is used to solve initial value ODE. The programs of methods are executed for different values of \(b\), and the results are presented in section 3.

3 Numerical Experiment
In this section some numerical experiments are presented. These experiments are performed by spectral method and FDM on two cases \(b = 0\) for \(T = 72h\), and \(b \neq 0\) for \(T = 12h\). The important parameters of problem are given as follows:

\(\varepsilon = 0.0001\), \(g = 0.1129\), \(k_1 = 0.0476\), \(k_2 = 0.3193\), \(\mu_{G_1} = 0\), \(\mu_{G_2} = 0\), \(\mu_S = 0\),

\(\mu_M = 0\), \(a = 100\), \(\theta = 0.05\). The case \(b = 0\) related to the influence of Taxel (anti-cancers drug), and \(b \neq 0\), corresponds with no anti-cancer drug.
3.1 Example
Consider \( b = 0 \), and \( T = 72h \), with above parameters the results are given for \( N = 100 \) and \( dt = 0.005 \) the numerical results for points \( x_i \) and \( t_j \) are given in Table 3.1 and, for spectral method or FDM. The graph of \( G_1 + G_2 + M + S \) is presented in Figure 1. The results can be compared with experimental data obtained by cytometry (Figure 3).

### Table 3.1. The results for example 3.1

<table>
<thead>
<tr>
<th>( i )</th>
<th>( j )</th>
<th>( x_i )</th>
<th>( t_j )</th>
<th>( G_1(x, t_j) )</th>
<th>( S(x, t_j) )</th>
<th>( G_2(x, t_j) )</th>
<th>( M(x, t_j) )</th>
</tr>
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<td>0</td>
<td>0.5</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>1100</td>
<td>0.073</td>
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</table>

![Figure 1: the graphs of \( S + M + G_1 + G_2 \) by FDM (a), and spectral method for \( b = 0 \) (b), and experimental result (c)](image)

3.2 Example
Consider \( b = 0.9692 \) and \( T = 12h \) with mentioned parameters the results are given for \( N = 100 \) and \( dt = 0.005 \) the numerical result for points \( x_i \) and \( t_j \) are given in Table 3.3 or Table 3.2 for spectral method and FDM, The sum of the solutions for \( G_1, S, G_2, \) and \( M \) are presented in Figure 2. The results are similar with experimental results which are obtained by cytometry (Figure 3).

### Table 3.2. The results for example 3.2

<table>
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<tr>
<th>( i )</th>
<th>( j )</th>
<th>( x_i )</th>
<th>( t_j )</th>
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<th>( S(x, t_j) )</th>
<th>( G_2(x, t_j) )</th>
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4 Conclusion
In this paper the mathematical model of in vitro human tumor cell population kinetic is simulated numerically by two methods: FDM and spectral method, the results are presented in section 3 with appropriate values of parameters that appear in the model. The results are evidently similar with experimental data. The results of the spectral method are compared with FDM, demonstrate less precision, that is the consequence of using implicit Euler method, instead of more powerful initial value method. However, the results show efficiency and reliability of the methods.

References

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