A new interval-valued approximation of interval-valued fuzzy numbers

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Abstract
In this paper, we proposed a new interval-valued approximation of interval-valued fuzzy numbers, which is the best one with respect to a certain measure of distance between interval-valued fuzzy numbers. Also, a set of criteria for interval-valued approximation operators is suggested.

Keywords: Fuzzy number, Interval-valued approximation, distance.

1 Introduction

Fuzzy numbers play significant role among all fuzzy sets since the predominant carrier of information are numbers.

As a generalization of an ordinary Zadeh fuzzy set, the notion of interval-valued fuzzy set was suggested for the first time by Gorzalczany[1] and Tursen[3]. Also, to this field, Wang and Li[6] defined interval-valued fuzzy numbers and gave their extended operations.

Hong et.al. continuous Wang’s work and expressed some algebraic properties of interval-valued fuzzy numbers. Also, a metric on interval-valued fuzzy numbers were proposed and investigate some convergence theorems for sequences of interval-valued fuzzy numbers with respect to the metric are treated.

However, sometimes we have to approximate a given fuzzy set by a crisp one. If we then use a defuzzification operator which replaces a fuzzy set by a single number we generally loose too many important information. Therefore, a crisp set approximation of a fuzzy set is often advisable.

In this paper, we proposed a new interval-valued distance to approximate interval-valued fuzzy numbers. To this end, we prove the metric properties of our proposed distance. Similar [7], we propose nearest interval-valued approximation of interval-valued fuzzy number with respect to proposed metric. Also, we extend the good-reasonable properties of interval-valued approximate operator and check their properties to our approximate operator.

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Our paper is organized as follows: in Section 2 we express some basic concepts and interval operations for interval-valued fuzzy numbers. In Section 3 we proposed Nearest interval-valued approximation with respect to new interval-valued metric. Additionally, we investigate some proposed criteria to present a good-reasonable interval-valued approximate operators.

2 Preliminaries

In this section, we gathered some needed definitions and theorems which will be applied in future sections. Also, we use the same notations that Wang and Li used in [6]. Let $I = [0, 1]$ and let $[I] = \{[a, b] \mid a \leq b, a, b \in I\}$. For any $a \in I$, define $\bar{a} = [a, a]$.

Definition 2.1. If $a_i \in I$, $t \in T$, then we define $\bigvee_{i \in T} a_i = \sup\{a_i : t \in T\}$ and $\bigwedge_{i \in T} a_i = \inf\{a_i : t \in T\}$. Also, is defined for $[a_i, b_i] \in [I], t \in T$

$$\bigvee_{i \in T} [a_i, b_i] = [\bigvee_{i \in T} a_i, \bigvee_{i \in T} b_i]$$

$$\bigwedge_{i \in T} [a_i, b_i] = [\bigwedge_{i \in T} a_i, \bigwedge_{i \in T} b_i]$$

Definition 2.2. Let $[a_1, b_1], [a_2, b_2] \in [I]$, then

$$[a_1, b_1] = [a_2, b_2] \iff a_1 = a_2, b_1 = b_2$$

$$[a_1, b_1] \leq [a_2, b_2] \iff a_1 \leq a_2, b_1 \leq b_2$$

$$[a_1, b_1] < [a_2, b_2] \iff [a_1, b_1] \leq [a_2, b_2], \text{ but, } a_1 \neq a_2, b_1 \neq b_2$$

Definition 2.3. Let $X$ be an ordinary nonempty set. Then the mapping $A : X \rightarrow [I]$ is called an interval-valued fuzzy set on $X$. All interval-valued fuzzy sets on $X$ are denoted by $IF_X$.

Definition 2.4. If $A \in IF(X)$, let $A(x) = [A^-(x), A^+(x)]$, where $x \in X$. Then two ordinary sets $A^-(X) \rightarrow I$ and $A^+(X) \rightarrow I$ are called lower fuzzy set and upper fuzzy set about $A$, respectively.

Definition 2.5. Let $A \in IF(X), [r_1, r_2] \in [I]$, then we called

$$A_{[r_1, r_2]} = \{x \in X \mid A^-(x) \geq r_1, A^+(x) \geq r_2\}$$

$$A_{(r_1, r_2)} = \{x \in X \mid A^-(x) > r_1, A^+(x) > r_2\}$$

the $[r_1, r_2]$-level set and $(r_1, r_2)$-level set of $A$ respectively. And let $A^- = \{x \in X \mid A^-(x) \geq r\}, A^+ = \{x \in X \mid A^+(x) \geq r\}$.

Definition 2.6. Let $A \in IF(R)$, i.e. $A : R \rightarrow I$. Assume the following conditions are satisfied:

1- $A$ is normal, i.e. there exists $x_0 \in R$ such that $A(x_0) = 1$.

2- For arbitrary $[r_1, r_2] \in [I] - \{0\}$, $A_{[r_1, r_2]}$ is closed bounded interval, then we call $A$ an interval-valued fuzzy number on real line $R$.

Let $IF^+(R)$ denote the set of all interval-valued fuzzy numbers on $R$, and write $[I]^+ = [I] - \{0\}$.

Definition 2.7. Let $A \in IF(R)$. The $A$ is called an interval convex set, if for any $x, y \in R$ and $\lambda \in [0, 1]$, we have

$$A(\lambda x + (1 - \lambda) y) \supseteq A(x) \cap A(y)$$

Definition 2.8. Let $A, B \in IF(R), \bullet \in \{+, -, \times, \div\}$, define their extended operations to $(A \bullet B)(z) = \bigvee_{x+y} (A(x) \cap B(y))$. For each $[r_1, r_2] \in [I]^+$ we write

$$A_{[r_1, r_2]} \bullet B_{[r_1, r_2]} = \{x \bullet y \mid x \in A_{[r_1, r_2]}, y \in B_{[r_1, r_2]}\}$$
Definition 2.9. Let $A \in IF^+(\mathbb{R})$, Then $A$ is called positive interval-valued fuzzy number, if $A(x) = 0$ whenever $x \leq 0$, and $A$ is called a negative interval-valued fuzzy number, if $A(x) = 0$ whenever $x \geq 0$.

All positive interval-valued fuzzy numbers and all negative interval-valued fuzzy numbers are denoted by $IF^+_+(\mathbb{R})$ and $IF^-_+(\mathbb{R})$, respectively.

Theorem 2.1. Let $A, B \in IF^+(\mathbb{R})$, $\bullet \in \{+, -, \cdot, \div\}$. Then

$$(A \bullet B)(z) = [(A^- \bullet B^-)(z), (A^\bullet B^\bullet)(z)]$$

Corollary 2.1. Let $A, B \in IF^+(\mathbb{R})$, for any $[r_1, r_2] \in [l]^+$,

$$(A \bullet B)_{[r_1, r_2]} = (A_{[r_1, r_2]} \bullet B_{[r_1, r_2]})$$

where, $\bullet \in \{+, -, \cdot, \div\}$, and $B \in IF^+_+(\mathbb{R}) \cup IF^-_+(\mathbb{R})$ whenever $\bullet$ chooses $\div$.

Definition 2.10. Let $A \in IF^+(\mathbb{R})$, then the power of $A$, $A^p$, where $p \in \mathbb{R}$ is denoted as

$$A^p(x) = [(A^-(x))^p, (A^+(x))^p]$$

where,

$$(A^-)^p = [(A_1^-)^p, (A_2^-)^p]$$

and

$$(A^+)^p = [(A_1^+)^p, (A_2^+)^p]$$

Definition 2.11. Let $A, B \in IF^+(\mathbb{R})$, then $(A \bullet B)^p$, where $\bullet \in \{+, -, \cdot, \div\}$ and $p \in \mathbb{R}$ is denoted as

$$(A \bullet B)^p = [(A^-)^p \bullet (B^-)^p, (A^+)^p \bullet (B^+)^p]$$

In this paper, r-cuts of lower fuzzy number $A^-(x)$ is expressed as $[A_1^-(r), A_2^-(r)]$, where

$$A_1^-(r) = \inf \{x \in \mathbb{R} : A^-(x) \geq r\}, \quad 0 \leq r \leq 1$$

$$A_2^-(r) = \sup \{x \in \mathbb{R} : A^-(x) \geq r\}, \quad 0 \leq r \leq 1$$

and the r-cut of upper fuzzy number $A^+(x)$ is expressed as $[A_1^+(r), A_2^+(r)]$, where

$$A_1^+(r) = \inf \{x \in \mathbb{R} : A^+(x) \geq r\}, \quad 0 \leq r \leq 1$$

$$A_2^+(r) = \sup \{x \in \mathbb{R} : A^+(x) \geq r\}, \quad 0 \leq r \leq 1$$

For two arbitrary interval-valued fuzzy numbers $A$ with r-cut representation $[A_1^-(r), A_2^+(r)]$, where $A^- = [A_1^-(r), A_2^-(r)]$ and $A^+ = [A_1^+(r), A_2^+(r)]$, interval-valued fuzzy number $B$ with r-cut representation $[B_1^-(r), B_2^+(r)]$, where $B^- = [B_1^-(r), B_2^-(r)]$ and $B^+ = [B_1^+(r), B_2^+(r)]$, the quantity

$$d_1(A, B) = [d_1^-, d_1^+]$$

where,

$$d_1^- = \left[\sqrt{\int_0^1 (A_1^-(r) - B_1^-(r))^2 + (A_2^-(r) - B_2^-(r))^2}\right],$$

$$d_1^+ = \left[\sqrt{\int_0^1 (A_1^+(r) - B_1^+(r))^2 + (A_2^+(r) - B_2^+(r))^2}\right],$$

is the distance between $A$ and $B$. 

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3 Nearest interval-valued approximation

Suppose that we want to approximate an interval-valued fuzzy number by an interval-valued interval. To this end, we have to use an interval operator $C : IF^2(\mathbb{R}) \rightarrow IP^2(\mathbb{R})$ where, $IP^2(\mathbb{R})$ is family of closed interval-valued intervals.

In this section, we will propose an interval-valued approximation operator called the nearest interval-valued interval approximation.

Suppose A is an interval-valued fuzzy number and its $r$-cut as $[A_L^r, A_U^r]$. Given A we will try to find a closed interval-valued interval $C_d(A)$ which is the nearest to A with respect to metric $d_l$. We can do it since each interval-valued interval is also an interval-valued fuzzy number with constant $r$-cuts for all $r \in (0,1]$. Hence, let $C_d(A) = [C_L, C_U]$, where

$$C_L = [C_L^1, C_L^2], \quad C_U = [C_U^1, C_U^2].$$

Now, we have to minimize $d_l(A, C_d(A))$ with respect to $C_L^1$, $C_L^2$, $C_U^1$ and $C_U^2$.

In order to minimize $d_l(A, C_d(A))$ it suffices to minimize function $D_l(A, C_d(A)) = d_l^2(A, C_d(A))$. Thus we have to find partial derivatives as follow:

$$\frac{\partial D_l(C_L, C_U)}{\partial C_L^i} = \left[-2 \int_0^1 A_L^i(r) - C_L^i(r), 0\right] = 0$$
$$\frac{\partial D_l(C_L, C_U)}{\partial C_U^i} = \left[0, -2 \int_0^1 A_U^i(r) - C_U^i(r)\right] = 0$$
$$\frac{\partial D_l(C_L, C_U)}{\partial C_L^i} = \left[-2 \int_0^1 A_L^i(r) - C_L^i(r), 0\right] = 0$$
$$\frac{\partial D_l(C_L, C_U)}{\partial C_U^i} = \left[0, -2 \int_0^1 A_U^i(r) - C_U^i(r)\right] = 0$$

The solution is

$$C_L^i = \int_0^1 A_L^i(r)dr, \quad C_U^i = \int_0^1 A_U^i(r)dr$$
$$C_L^i = \int_0^1 A_L^i(r)dr, \quad C_U^i = \int_0^1 A_U^i(r)dr$$

Moreover, since

$$\det \left[\frac{\partial^2 D_l(C_L, C_U)}{\partial C_L^i \partial C_U^j}\right] = 16 > 0, \quad i + j = 4, \quad i, j = 0, \ldots, 4.$$ 

and $\frac{\partial^4 D_l(C_L, C_U)}{\partial C_L^i \partial C_U^j} > 0$, then $C_L$ and $C_U$ given by $C_L = [C_L^1, C_L^2]$ and $C_U = [C_U^1, C_U^2]$, actually minimize $D_l(C_L, C_U)$ and simultaneously, minimize $d_l(A, C_d(A))$. Therefore the interval-valued interval

$$C_d(A) = \left[\int_0^1 A_L^i(r)dr, \int_0^1 A_U^i(r)dr\right]$$

is indeed the nearest interval-valued approximation of interval-valued fuzzy number A with respect to metric $d_l$.

3.1 Good-reasonable properties of interval-valued interval approximation

In this subsection, we proposed some good-reasonable properties of interval-valued approximation of interval-valued fuzzy numbers as criteria for approximation, similar works which is explained and discussed by Grzegorzewski et.al.[8].
3.1 Translation Invariance:

We say that an interval-valued approximation operator \( C_{d_t} \) is translation invariance if \( C_{d_t}(A + \zeta) = C_{d_t}(A) + \zeta \), for all \( \zeta \in \mathbb{R} \).

**Proposition 3.1.** Our proposed interval-valued approximation \( C_{d_t}(\cdot) \) is not translation invariance.

**Proof.** Let \( A \in IF^+(\mathbb{R}) \), \( \zeta \in \mathbb{R} \) then

\[
C_{d_t}(A + \zeta) = \left[ \int_0^1 ((A(r) + \zeta)_1)^2 + ((A(r) + \zeta)_2)^2, \int_0^1 ((A(r) + \zeta)_1)^2 + ((A(r) + \zeta)_2)^2 \right] =
\]

\[
\left[ \int_0^1 (A^-_1(r))^2 + (A^-_2(r))^2, \int_0^1 (A^+_1(r))^2 + (A^+_2(r))^2 \right] + 2[\zeta^2, \zeta^2] = C_{d_t} + 2\zeta^2
\]

Thus we have

\[
C_{d_t}(A + \zeta) = C_{d_t}(A) + 2\zeta^2 \implies C_{d_t}(A + \zeta) = C_{d_t}(A) + \sqrt{2}\zeta^2
\]

\( \square \)

3.1.2 Scale invariant:

We say that an interval-valued approximation operator \( C_{d_t} \) is scale invariant if \( C_{d_t}(\lambda A) = \lambda C_{d_t}(A) \), for all \( \lambda \in \mathbb{R} - \{0\} \).

**Proposition 3.2.** Our proposed interval-valued approximation \( C_{d_t}(\cdot) \) is scale invariant.

**Proof.** Let \( A \in IF^+(\mathbb{R}) \), then

\[
C_{d_t}(\lambda A) = \left[ \int_0^1 (\lambda A^-_1)^2 + (\lambda A^-_2)^2, \int_0^1 (\lambda A^+_1)^2 + (\lambda A^+_2)^2 \right] =
\]

\[
\lambda^2 \left[ \int_0^1 (A^-_1)^2 + (A^-_2)^2, \int_0^1 (A^+_1)^2 + (A^+_2)^2 \right] =
\]

\[
\lambda^2 C_{d_t}(A)
\]

Thus, \( C_{d_t}(\lambda A) = \lambda C_{d_t}(A) \).

\( \square \)

3.1.3 Monotony:

The criterion of monotony states that for any two interval-valued fuzzy numbers \( A \) and \( B \) holds

\[
if \ A \subseteq B \ then \ C_{d_t}(A) \subseteq C_{d_t}(B)
\]

**Proposition 3.3.** Our proposed interval-valued approximation \( C_{d_t}(\cdot) \) is monotone.

**Proof.** Let \( A, B \in IF^+(\mathbb{R}) \) and \( A \subseteq B \) then there exists a function \( \Theta(r) = [\Theta_1, \Theta_2] \geq 0 \) and \( \Psi(r) = [\Psi_1, \Psi_2] \geq 0 \) such that

\[
A^- (r) = B^- (r) + \Theta (r) \equiv A^-_1 (r) = B^-_1 (r) + \Theta_1 (r), \text{ } A^-_2 (r) = B^-_2 (r) + \Theta_2 (r)
\]

\[
A^+ (r) = B^+ (r) - \Psi (r) \equiv A^+_1 (r) = B^+_1 (r) - \Psi_1 (r), \text{ } A^+_2 (r) = B^+_2 (r) - \Psi_2 (r)
\]

(3.3)

and by using definition distance \( C_{d_t}(A) \) we have

\[
C_{d_t}(A) = \left[ \int_0^1 A^-_1 (r)dr, \int_0^1 A^-_2 (r)dr \right], \int_0^1 A^+_1 (r)dr, \int_0^1 A^+_2 (r)dr
\]
then by substitution of equivalent values from (3.3) we have two inequalities as follow:

\[ \int_{0}^{1} [B_{1}^{-}(r) + \Theta_{1}(r)]^2 dr + \int_{0}^{1} [B_{2}^{-}(r) + \Theta_{2}(r)]^2 dr \geq \int_{0}^{1} [B_{1}^{-}(r)]^2 dr + \int_{0}^{1} [B_{2}^{-}(r)]^2 dr, \]

\[ \int_{0}^{1} [B_{1}^{+}(r) - \Psi_{1}(r)]^2 dr + \int_{0}^{1} [B_{2}^{+}(r) - \Psi_{2}(r)]^2 dr \leq \int_{0}^{1} [B_{1}^{+}(r)]^2 dr + \int_{0}^{1} [B_{2}^{+}(r)]^2 dr, \]

or equivalently, is obtained

\[ [C_A]^a \geq [C_B]^a, \quad [C_A]^u \leq [C_B]^u \]

Therefore, \( C_{D_l}(A) \subseteq C_{D_l}(B) \) and should be \( C_{d_l}(A) \subseteq C_{d_l}(B) \).

3.1.4 Continuity:

An interval-valued interval approximation \( C_{d_l}(A) \) is called continuous if for any \( A, B \in IF^+ \) we have

\[ \forall \varepsilon, \exists \delta : d_l[A, B] < \delta \implies d_l[C_{d_l}(A), C_{d_l}(B)] < \varepsilon \]

where, \( d_l : IF^+(\mathbb{R}) \times IF^+(\mathbb{R}) \rightarrow [\mathbb{R}^+, \mathbb{R}^+] \) denotes a metric defined on the family of all interval-valued fuzzy numbers.

Proposition 3.4. : Our proposed interval-valued approximation \( C_{d_l}(.) \) is continuous.

Proof. Let \( A, B \in IF^+(\mathbb{R}) \) then

\[ D_l[C_{d_l}(A), C_{d_l}(B)] = \]

\[ \left[ \int_{0}^{1} \{(C_A^1)_{\lambda} - (C_B^1)_{\lambda}\}^2 + \{(C_A^2)_{\lambda} - (C_B^2)_{\lambda}\}^2 \right]^{1/2} \]

\[ \left[ \left( \int_{0}^{1} (A_1^+ - B_1^+) \right)^2 + \left( \int_{0}^{1} (A_2^+ - B_2^+) \right)^2 \right] \leq \]

\[ \left[ \int_{0}^{1} (A_1^- - B_1^-)^2 + \int_{0}^{1} (A_2^- - B_2^-)^2 \right] \]

or equivalently, \( d_l[C_{d_l}(A), C_{d_l}(B)] \leq d_l[A, B] \).

3.2 Metric Properties

In this part, we prove that our proposed interval-valued distance \( d_l \) has metric properties.

Metric properties of \( d_l \):

1- \( d_l[A, B] \geq 0 \),

2- \( d_l[A, B] = d_l(B, A) \),

3- \( d_l[A, C] \leq d_l[A, B] + d_l[B, C] \),

4- \( d_l[A, B] = 0 \implies A = B \).

Let \( A, B, C \in IF^+(\mathbb{R}) \), then case(1) and case(2) by applying definition of \( d_l \) is obvious. Thus we proof case(3) and case(4).
proof case (3):

\[ D_I[A, C] = \left[ \int_0^1 (A_1^- - C_1^-)^2 + (A_2^- - C_2^-)^2, \int_0^1 (A_1^+ - C_1^+)^2 + (A_2^+ - C_2^+)^2 \right] = \]
\[ \left[ \int_0^1 (A_1^- - B_1^- - B_1^-)^2 + (A_2^- - B_2^- - B_2^-)^2, \int_0^1 (A_1^+ - B_1^+ - B_1^+)^2 + (A_2^+ - B_2^+ - B_2^+)^2 \right] = \]
\[ \int_0^1 (A_1^+ - B_1^+)^2 + (A_2^+ - B_2^+)^2 + (A_2^- - C_2^-)^2 + (B_2^- - C_2^-)^2 \right] = \]

\[ d_I[A, C] \leq d_I[A, B] + d_I[B, C] \]

proof case (4):

Let \( d_i[A, B] = [d_i^-, d_i^+] = 0 \) then,

\[ d_i^- = \left\lfloor \int_0^1 (A_1^-(r) - B_1^-(r))^2 + (A_2^-(r) - B_2^-(r))^2 \right\rfloor = 0, \]
\[ d_i^+ = \left\lceil \int_0^1 (A_1^+(r) - B_1^+(r))^2 + (A_2^+(r) - B_2^+(r))^2 \right\rfloor = 0, \]

Therefore, we have

\[ \int_0^1 (A_1^-(r) - B_1^-(r))^2 + (A_2^-(r) - B_2^-(r))^2 = 0, \]
\[ \int_0^1 (A_1^+(r) - B_1^+(r))^2 + (A_2^+(r) - B_2^+(r))^2 = 0. \]

Hence

\[ (A_1^-(r) - B_1^-(r))^2 = 0, \quad (A_2^-(r) - B_2^-(r))^2 = 0 \]
\[ (A_1^+(r) - B_1^+(r))^2 = 0, \quad (A_2^+(r) - B_2^+(r))^2 = 0 \]

So, should have

\[ A_1^-(r) = B_1^-(r), \quad A_2^-(r) = B_2^-(r), \quad A_1^+(r) = B_1^+(r), \quad A_2^+(r) = B_2^+(r) \]

Therefore, \( A = B \) and the proof is complete.

Also, our proposed interval-valued approximate operator is coincide with interval approximate operator where, fuzzy numbers are replaced with interval-valued fuzzy numbers. So, if

\[ A_v = A_v^+, \quad B_v = B_v^+ \]

then

\[ d_i^- = d_i^+ = \sqrt{\int_0^1 (A_v^- - B_v^-)^2 + (A_v^+ - B_v^+)^2} \]
4 Conclusion

In this paper, we proposed a new interval-valued approximation of interval-valued fuzzy numbers, which is the best one with respect to a new measure of distance between interval-valued fuzzy numbers. Also, a set of criteria for interval-valued approximation operators is suggested. Also, we see that our proposed interval-valued distance is coincide with ordinary distance where fuzzy number are replaced with interval-valued fuzzy number. For future work we want to extend trapezoidal approximation to interval-valued fuzzy numbers.

References


