Common Fixed Point Theorems of $C$-distance on Cone Metric Spaces

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Abstract

Wang and Guo (2011) defined the $c$-distance in a cone metric space and used this concept to prove a common fixed point theorem. In this paper, we extend and generalize the main results of Wang and Guo and also give the applications of our theorem as shown in the examples.

Keywords: Cone metric space; $c$-distance; Common fixed point.

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1 Introduction

Since the Banach’s contraction mapping principle very usefulness, it has become a popular tool in solving many problems in mathematical analysis. For some more results of generalization of this principle, refer to [5, 9, 10, 11, 12, 13, 14] and references mentioned therein. In 2007, Huang and Zhang [3] first introduced the concept of cone metric spaces. Cone metric spaces is a some generalize version of metric spaces, where each pair of points is assigned to a member of a real Banach space over the cone. They also established and proved the existence of fixed point theorem which is an extension of the Banach’s contraction mapping principle to cone metric spaces. Later, researchers have studied and extend fixed point problems in cone metric spaces (see [1, 2, 6, 7, 8, 15, 17]). Recently, Wang and Guo [16] introduced the concept of the $c$-distance in a cone metric spaces, which is a cone.

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version of $w$-distance of Kada et al. [4], and established common fixed point theorem in cone metric spaces which is more general than the classical Banach’s contraction mapping principle. We state Theorem 2.2 of Wang and Guo in [16] as follows:

**Theorem 1.1.** Let $(X, d)$ be a cone metric space, $P$ be a normal cone with normal constant $K$, $q$ be a $c$-distance on $X$ and $S, T : X \rightarrow X$ such that $T(X) \subseteq S(X)$ and $S(X)$ be a complete subspace of $X$. Suppose that there exists a positive real numbers $a_1, a_2, a_3, a_4$ with $a_1 + 2a_2 + a_3 + a_4 < 1$ such that

$$q(Tx, Ty) \preceq a_1q(Sx, Sy) + a_2q(Sx, Ty) + a_3q(Sx, Tx) + a_4q(Sy, Ty)$$

for all $x, y \in X$. If $S$ and $T$ satisfy

$$\inf\{\|q(Tx, y)\| + \|q(Sx, y)\| + \|q(Sx, Tx)\| : x \in X\} > 0$$

for all $y \in X$ with $Ty \neq y$ or $Sy \neq y$, then $S$ and $T$ have a common fixed point in $X$.

In this paper, we develop and generalize the common fixed point theorem on $c$-distance of Wang and Guo [16]. We also apply our results to conclude the existence of common fixed point theorem in the example. Main theorem extends and improves Theorem 3.2 and many works in fixed point theory in the literature.

## 2 Preliminaries

Let $(E, \| \cdot \|)$ be a real Banach space and $\theta$ denote the zero element in $E$. A cone $P$ is a subset of $E$ such that

1. $P$ is nonempty closed set and $P \neq \{\theta\}$;
2. if $a, b$ are nonnegative real numbers and $x, y \in P$, then $ax + by \in P$;
3. $P \cap (-P) = \{\theta\}$.

For any cone $P \subseteq E$, the partial ordering $\preceq$ with respect to $P$ defined by

$$x \preceq y \iff y - x \in P.$$

We shall write $x < y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}(P)$, where $\text{int}(P)$ denotes the interior of $P$. A cone $P$ is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \implies \|x\| \leq K\|y\|.$$

The least positive number satisfying above is called the normal constant of $P$.

**Remark 2.1.** [3], Let $E$ be a real Banach space with a cone $P$. If $x \preceq kx$, where $x \in P$ and $0 < k < 1$, then $x = \theta$.

**Lemma 2.1.** If $E$ is a real Banach space with a cone $P$, then if $\theta \preceq x \preceq y$ and $k$ is a nonnegative real number, then $\theta \preceq kx \preceq ky$.

Using the notation of a cone, we have following definitions of cone metric space.
Definition 2.1. [3], Let $X$ be a nonempty set and $E$ be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone $P \subseteq E$. Suppose that the mapping $d : X \times X \to E$ satisfies the following conditions:

1. $\theta < d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$ and $(X, d)$ is called a cone metric space.

Definition 2.2. [3], Let $(X, d)$ be a cone metric space. Let $\{x_n\}$ be a sequence in $X$ and $x \in X$.

1. If for every $c \in E$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to $x$, and $x$ is the limit of $\{x_n\}$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
2. If for every $c \in E$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that for all $m, n > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in $X$.
3. If every Cauchy sequence in $X$ is convergent, then $X$ is called a complete cone metric space.

For other basic properties on cone metric space, the reader can read in [3]. Next, we give the notion of $c$-distance on a cone metric space $(X, d)$ of Wang and Guo in [16], which is a generalization of $w$-distance of Kada et al. [4] and some properties.

Definition 2.3. [16], Let $(X, d)$ be a cone metric space. Then a function $q : X \times X \to E$ is called a $c$-distance on $X$ if the following are satisfied:

(q1) $\theta \leq q(x, y)$ for all $x, y \in X$;
(q2) $q(x, z) \leq q(x, y) + q(y, z)$ for all $x, y, z \in X$;
(q3) for each $x \in X$ and $n \geq 1$, if $q(x, y_n) \leq u$ for some $u = u_x \in P$, then $q(x, y) \leq u$ whenever $\{y_n\}$ is a sequence in $X$ converging to a point $y \in X$;
(q4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll e$.

Remark 2.2. The $c$-distance $q$ is a $w$-distance on $X$ if we take $(X, d)$ is a metric space, $E = \mathbb{R}^+$, $P = [0, \infty)$ and $(q3)$ is replaced by this condition: For any $x \in X$, $q(x, \cdot) : X \to \mathbb{R}^+$ is lower semi-continuous. Moreover, $(q3)$ holds whenever $q(x, \cdot)$ is lower semi-continuous. Thus if $(X, d)$ is a metric space, $E = \mathbb{R}^+$, $P = [0, \infty)$, then every $w$-distance is a $c$-distance. But the converse is not true in general case. Therefore, the $c$-distance is a generalization of the $w$-distance.

Example 2.1. [16], Let $(X, d)$ be a cone metric space and $P$ be a normal cone. Define a mapping $q : X \times X \to E$ by

$$q(x, y) = d(x, y)$$

for all $x, y \in X$. Then $q$ is $c$-distance.
Example 2.2. [16], Let $(X, d)$ be a cone metric space and $P$ be a normal cone. Define a mapping $q : X \times X \to E$ by
\[ q(x, y) = d(u, y) \]
for all $x, y \in X$, where $u$ is a fixed point in $X$. Then $q$ is $c$-distance.

Example 2.3. [16], Let $E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and
\[ P = \{ x \in E : x(t) \geq 0 \text{ on } [0, 1] \} \]
(this cone is not normal). Let $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ by
\[ d(x, y) = |x - y|\varphi \]
for all $x, y \in X$, where $\varphi : [0, 1] \to \mathbb{R}$ such that $\varphi(t) = e^t$. Then $(X, d)$ is a cone metric space. Define a mapping $q : X \times X \to E$ by
\[ q(x, y) = (x + y)\varphi \]
for all $x, y \in X$. Then $q$ is $c$-distance.

Example 2.4. [16], Let $E = \mathbb{R}$ and $P = \{ x \in E : x \geq 0 \}$. Let $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ by
\[ d(x, y) = |x - y| \]
for all $x, y \in X$. Then $(X, d)$ is a cone metric space. Define a mapping $q : X \times X \to E$ by
\[ q(x, y) = y \]
for all $x, y \in X$. Then $q$ is $c$-distance.

Remark 2.3. On $c$-distance $q(x, y) = q(y, x)$ does not necessarily hold and $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

Next lemma is useful for the main result in this paper.

Lemma 2.2. [16], Let $(X, d)$ be a cone metric space, $q$ be a $c$-distance on $X$ and $\{x_n\}$ be a sequence in $X$. If there exists the sequence $\{u_n\}$ in $P$ converging to $\theta$ and
\[ q(x_n, x_m) \leq u_n \]
for $m > n$, then $\{x_n\}$ is a Cauchy sequence in $X$.

3 Common fixed point theorems on $c$-distance

The following is the main result of this paper. We prove a common fixed point theorem by using the idea of $c$-distance. Our theorem extends the contractive condition of Theorem 1.1 from constant real numbers to some control functions.

Theorem 3.1. Let $(X, d)$ be a cone metric space, $P$ be a normal cone with normal constant $K$, $q$ be a $c$-distance on $X$ and $S, T : X \to X$ such that $T(X) \subseteq S(X)$ and $S(X)$ be a complete subspace of $X$. Suppose that there exists mappings $\alpha, \beta, \gamma, \mu : X \to [0, 1)$ such that the following assertions hold:
1. \( \alpha(Tx) \leq \alpha(Sx), \beta(Tx) \leq \beta(Sx), \gamma(Tx) \leq \gamma(Sx) \) and \( \mu(Tx) \leq \mu(Sx) \) for all \( x \in X \);
2. \( (\alpha + 2\beta + \gamma + \mu)(x) < 1 \) for all \( x \in X \);
3. \( q(Tx, Ty) \leq \alpha(Sx)q(Sx, Sy) + \beta(Sx)q(Sx, Ty) + \gamma(Sx)q(Sx, Tx) + \mu(Sx)q(Sy, Ty) \)
   for all \( x, y \in X \);
4. \( \inf\{\|q(Tx, y)\| + \|q(Sx, y)\| + \|q(Sx, Tx)\| : x \in X\} > 0 \) for all \( y \in X \) with \( Ty \neq y \) or \( Sy \neq y \).

Then \( S \) and \( T \) have a common fixed point in \( X \).

**Proof.** Let \( x_0 \) be an arbitrary point in \( X \). Since \( T(X) \subseteq S(X) \), there exists a point \( x_1 \in X \) such that \( Tx_0 = Sx_1 \). By induction, we construct the sequence \( \{x_n\} \) in \( X \) such that

\[
Sx_n = Tx_{n-1}
\]

for all \( n \geq 1 \). From (1) and (3), we have

\[
q(Sx_n, x_{n+1}) = q(Tx_{n-1}, Tx_n) \\
\leq \alpha(Sx_{n-1})q(Sx_{n-1}, x_n) + \beta(Sx_{n-1})q(Sx_{n-1}, Tx_n) \\
+ \gamma(Sx_{n-1})q(Sx_{n-1}, Tx_{n-1}) + \mu(Sx_{n-1})q(x_n, Tx_n) \\
= \alpha(Tx_{n-2})q(Sx_{n-1}, x_n) + \beta(Tx_{n-2})q(Sx_{n-1}, Sx_{n+1}) \\
+ \gamma(Tx_{n-2})q(Sx_{n-1}, x_n) + \mu(Tx_{n-2})q(Sx_{n}, Sx_{n+1}) \\
\leq \alpha(Sx_{n-2})q(Sx_{n-1}, x_n) + \beta(Sx_{n-2})q(Sx_{n-1}, Sx_{n+1}) \\
+ \gamma(Sx_{n-2})q(Sx_{n-1}, x_n) + \mu(Sx_{n-2})q(x_n, x_{n+1}) \\
\vdots
\]

which implies that

\[
q(Sx_n, x_{n+1}) \leq \left( \frac{\alpha(Sx_0) + \beta(Sx_0) + \gamma(Sx_0)}{1 - \beta(Sx_0) - \mu(Sx_0)} \right) q(x_{n-1}, x_n)
\]

(3.2)

for all \( n \geq 1 \). Now, we let

\[
k := \left( \frac{\alpha(Sx_0) + \beta(Sx_0) + \gamma(Sx_0)}{1 - \beta(Sx_0) - \mu(Sx_0)} \right) < 1.
\]
By repeating (3.3), we get
\[ q(Sx_n, Sx_{n+1}) \leq k^n q(Sx_0, Sx_1). \] 
(3.4)

Now, for positive integer \( m \) and \( n \) with \( m > n \geq 1 \), it follows from (3.4) that
\[ q(Sx_n, Sx_m) \leq q(Sx_n, Sx_{n+1}) + q(Sx_{n+1}, Sx_{n+2}) + \cdots + q(Sx_{m-1}, Sx_m) \]
\[ \leq k^n q(Sx_0, Sx_1) + k^{n+1} q(Sx_0, Sx_1) + \cdots + k^{m-1} q(Sx_0, Sx_1) \]
(3.5)
\[ \leq \left( \frac{k^n}{1-k} \right) q(Sx_0, Sx_1). \]

Since \( 0 \leq k < 1 \), we get \( \left( \frac{k^n}{1-k} \right) q(Sx_0, Sx_1) \to \theta \) as \( n \to 0 \). From Lemma 2.2, we have \( \{Sx_n\} \) is a Cauchy sequence in \( S(X) \). Therefore, there exists a point \( z \in S(X) \) such that \( Sx_n \to z \) as \( n \to \infty \) by the completeness of \( S(X) \). From (3.5) and (q3), we have
\[ q(Sx_n, z) \leq \left( \frac{k^n}{1-k} \right) q(Sx_0, Sx_1) \] 
(3.6)

for all \( n \geq 1 \). By \( P \) is a normal cone with constant \( K \) and (3.5), we get
\[ \|q(Sx_n, Sx_m)\| \leq K \left( \frac{k^n}{1-k} \right) \|q(Sx_0, Sx_1)\| \] 
(3.7)

for all \( m > n \geq 1 \). From (3.6), we get
\[ \|q(Sx_n, z)\| \leq K \left( \frac{k^n}{1-k} \right) \|q(Sx_0, Sx_1)\| \] 
(3.8)

for all \( n \geq 1 \).

Suppose that \( Tz \neq z \) or \( Sz \neq z \). Then by hypothesis, (3.7) and (3.8), we get
\[ 0 < \inf\{\|q(Tx, z)\| + \|q(Sx, z)\| + \|q(Sx, Tx)\| : x \in X\} \]
\[ \leq \inf\{\|q(Tx_n, z)\| + \|q(Sx_n, z)\| + \|q(Sx_n, Tx_n)\| : n \geq 1\} \]
\[ = \inf\{\|q(Sx_{n+1}, z)\| + \|q(Sx_n, z)\| + \|q(Sx_n, Sx_{n+1})\| : n \geq 1\} \]
\[ \leq \inf\left\{ K \left( \frac{k^{n+1}}{1-k} \right) \|q(Sx_0, Sx_1)\| + K \left( \frac{k^n}{1-k} \right) \|q(Sx_0, Sx_1)\| \right\} \]
\[ + K \left( \frac{k^n}{1-k} \right) \|q(Sx_0, Sx_1)\| : n \geq 1 \}
\[ = 0, \]
which is a contradiction. Therefore, we can conclude that \( z = Sz = Tz \). \( \square \)

**Corollary 3.1.** Let \((X,d)\) be a complete cone metric space, \( P \) be a normal cone with normal constant \( K \), \( q \) be a \( c \)-distance on \( X \) and \( T : X \to X \). Suppose that there exists mappings \( \alpha, \beta, \gamma, \mu : X \to [0,1) \) such that the following assertions hold:
1. \( \alpha(Tx) \leq \alpha(x), \beta(Tx) \leq \beta(x), \gamma(Tx) \leq \gamma(x) \) and \( \mu(Tx) \leq \mu(x) \) for all \( x \in X \);
2. \( (\alpha + 2\beta + \gamma + \mu)(x) < 1 \) for all \( x \in X \);
3. \( q(Tx, Ty) \leq \alpha(x)q(x, y) + \beta(x)q(x, Ty) + \gamma(x)q(x, Tx) + \mu(x)q(y, Ty) \) for all \( x, y \in X \);
4. \( \inf\{\|q(Tx, y)\| + \|q(x, y)\| + \|q(Tx, Tx)\| : x \in X\} > 0 \) for all \( y \in X \) with \( Ty \neq y \).

Then \( T \) has a fixed point in \( X \).

**Proof.** Take mapping \( S \) in Theorem 3.1 as \( I_X \) where \( I_X \) is an identity mapping on \( X \). \( \square \)

**Corollary 3.2.** [16]. Let \( (X, d) \) be a cone metric space, \( P \) be a normal cone with normal constant \( K \), \( q \) be a \( c \)-distance on \( X \) and \( S, T : X \to X \) such that \( T(X) \subseteq S(X) \) and \( S(X) \) be a complete subspace of \( X \). Suppose that there exists a positive real numbers \( a_1, a_2, a_3, a_4 \) with \( a_1 + 2a_2 + a_3 + a_4 < 1 \) such that

\[
q(Tx, Ty) \leq a_1q(Sx, Sy) + a_2q(Sx, Ty) + a_3q(Sx, Tx) + a_4q(Sy, Ty)
\]

for all \( x, y \in X \). If \( S \) and \( T \) satisfy

\[
\inf\{\|q(Tx, y)\| + \|q(x, y)\| + \|q(Sx, Tx)\| : x \in X\} > 0
\]

for all \( y \in X \) with \( Ty \neq y \) or \( Sy \neq y \), then \( S \) and \( T \) have a common fixed point in \( X \).

**Proof.** Take mappings \( \alpha, \beta, \gamma, \mu : X \to [0, 1) \) by \( \alpha(x) = a_1, \beta(x) = a_2, \gamma(x) = a_3 \) and \( \mu(x) = a_4 \) in Theorem 3.1 and then it follows from Theorem 3.1. \( \square \)

### 4 Some Examples

Next, we give some examples to illustrate the main result.

**Example 4.1.** Let \( E = \mathbb{R} \) and \( P = \{x \in E : x \geq 0\} \). Let \( X = [0, 1) \) and define a mapping \( d : X \times X \to E \) by

\[
d(x, y) = |x - y|
\]

for all \( x, y \in X \). Then \( (X, d) \) is a cone metric space. Define a mapping \( q : X \times X \to E \) by

\[
q(x, y) = 2d(x, y)
\]

for all \( x, y \in X \). Then \( q \) is \( c \)-distance. In fact, (q1)-(q3) are immediate. Let \( c \in E \) with \( 0 \ll c \) and put \( e = c/2 \). If \( q(z, x) \ll e \) and \( q(z, y) \ll e \), then we have

\[
d(x, y) \leq 2d(x, y) = 2|x - y| \leq 2|x - z| + 2|z - y| = q(z, x) + q(z, y) \ll e + e = c.
\]

This shows that (q4) holds. Therefore \( q \) is \( c \)-distance.

Let \( S, T : X \to X \) defined by \( S(x) = x \) and \( T(x) = \frac{x^2}{16} \) for all \( x \in X \). Take mappings \( \alpha, \beta, \gamma, \mu : X \to [0, 1) \) by

\[
\alpha(x) = \beta(x) = \frac{x + 1}{16}, \gamma(x) = \frac{2x + 3}{16} \quad \text{and} \quad \mu(x) = \frac{3x + 2}{16}
\]

for all \( x \in X \). Observe that
\(\alpha(Tx) = \beta(Tx) = \frac{1}{16}(x^2 + 1) \leq \frac{1}{16}(x^2 + 1) \leq \frac{x+1}{16} = \alpha(Sx) = \beta(Sx) \) for all \(x \in X\).

\(\gamma(Tx) = \frac{1}{16}\left(2x^2 + 3\right) \leq \frac{1}{16}(2x^2 + 3) \leq \frac{2x+3}{16} = \gamma(Sx) \) for all \(x \in X\).

\(\mu(Tx) = \frac{1}{16}\left(3x^2 + 2\right) \leq \frac{1}{16}(3x^2 + 2) \leq \frac{3x+2}{16} = \mu(Sx) \) for all \(x \in X\).

\((\alpha + 2\beta + \gamma + \mu)(x) = \frac{x+1}{16} + 2\left(\frac{x}{16}\right) + \frac{2x+3}{16} + \frac{3x+2}{16} = \frac{8x+8}{16} < 1 \) for all \(x \in X\).

For all \(x, y \in X\), we have
\[
q(Tx, Ty) = 2\left|\frac{x^2}{16} - \frac{y^2}{16}\right| \\
\leq \frac{2|x+y||x-y|}{16} \\
= \left(\frac{x+y}{16}\right)2|x-y| \\
\leq \left(\frac{x+1}{16}\right)2|x-y| \\
\leq \alpha(x)q(Sx, Sy) + \beta(x)q(Sx, Ty) + \gamma(x)q(Sx, Tx) + \mu(x)q(Sy, Ty).
\]

(6) For any \(y \neq Ty\), i.e., \(y > 0\), we get
\[
\inf\{\|q(Tx, y)\| + \|q(Sx, y)\| + \|q(Sx, Tx)\| : x \in X\} = 4\left|y - \frac{y^2}{16}\right| > 0.
\]

Therefore, all the conditions of Theorem 3.1 are satisfied. Thus we can conclude that \(S\) and \(T\) have a common fixed point in \(X\). This common fixed point is \(x = 0\).

Example 4.2. Let \(E = \mathbb{R}\) and \(P = \{x \in E : x \geq 0\}\). Let \(X = [0, 1)\) and define a mapping \(d : X \times X \to E\) by
\[
d(x, y) = |x - y|
\]
for all \(x, y \in X\). Then \((X, d)\) is a cone metric space. Define a \(c\)-distance \(q : X \times X \to E\) as in Example 4.1.

Let \(S, T : X \to X\) defined by \(S(x) = x\) and \(T(x) = \frac{x^2}{3}\) for all \(x \in X\). Take mappings \(\alpha, \beta, \gamma, \mu : X \to [0, 1)\) by
\[
\alpha(x) = \frac{2x+1}{3} \text{ and } \beta(x) = \gamma(x) = \mu(x) = 0
\]
for all \(x \in X\). Then we have the following statement holds:

(1) \(\alpha(Tx) = \frac{1}{3}\left(\frac{2x^2}{3} + 1\right) \leq \frac{1}{3}(2x^2 + 1) \leq \frac{2x+1}{3} = \alpha(Sx) \) for all \(x \in X\);

(2) \(\beta(Tx) = \gamma(Tx) = \mu(Tx) = 0 \leq 0 = \beta(Sx) = \gamma(Sx) = \mu(Sx) \) for all \(x \in X\);
(3) \((\alpha + 2\beta + \gamma + \mu)(x) = \frac{2x + 1}{3} < 1\) for all \(x \in X\);

(4) For each \(x, y \in X\), we have

\[
q(Tx, Ty) = 2\left|\frac{x^2}{4} - \frac{y^2}{4}\right|
\leq \frac{2|x+y||x-y|}{3}
= \left(\frac{x+y}{3}\right)2|x-y|
\leq \left(\frac{2x+y}{3}\right)2|x-y|
\leq \left(\frac{2x+1}{3}\right)2|x-y|
\leq \alpha(x)q(Sx, Sy) + \beta(x)q(Sx, Ty) + \gamma(x)q(Sx, Tx) + \mu(x)q(Sy, Ty);
\]

(6) For any \(y \neq Ty\), i.e., \(y > 0\), we get

\[
\inf\{\|q(Tx, y)\| + \|q(Sx, y)\| + \|q(Sx, Tx)\| : x \in X\} = 4\left|y - \frac{y^2}{3}\right| > 0.
\]

Therefore, the hypothesis of Theorem 3.1 is satisfied and so we can conclude that \(S\) and \(T\) have a common fixed point in \(X\). This common fixed point is \(x = 0\).

**Remark 4.1.** Although, Theorem 1.1 is essential tool in the cone metric \(d\) to conclude the existence of fixed points and common fixed points of some mappings. Sometimes the constant numbers which satisfy Theorem 1.1 is very difficult to find. Therefore, it is the most interest to define such mappings \(\alpha, \beta, \gamma, \mu\) as another auxiliary tool of the cone metric \(d\).

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