Common Fixed Points for Mappings in $G$-Cone Metric Spaces

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Abstract
The object of this paper is to obtain sufficient conditions for existence of common fixed points for three self mappings satisfying various contractive conditions in $G$-cone metric spaces. Our results will generalize and extend some recent results in fixed point theory.

Keywords: $G$-cone metric space; weakly compatible; point of coincidence; common fixed point.

1 Introduction

Over the past two decades a considerable amount of research work for the improvement of fixed point theory have executed by several authors. There has been a number of generalizations of the usual notion of metric spaces such as Gähler [6, 7] (called 2-metric spaces) and by Dhage [4, 5] (called $D$-metric spaces). However, Mustafa and Sims in [13] have pointed out that most of the results claimed by Dhage and others in $D$-metric spaces are invalid. To overcome these fundamental flaws, they introduced a new concept of generalized metric space called $G$-metric space [11] and obtained several interesting fixed point results in this structure. Another such generalization initiated by Huang and Zhang [9], replacing the set of real numbers by an ordered Banach space, called cone metric space and gave some fixed point theorems for contractive type mappings in a normal cone metric space. In [16], Rezapour and Hambarani omitted the assumption of normality in cone metric space, which is a milestone in developing fixed point theory in cone metric space. The study of existence of points of coincidence and fixed points for mappings in normal and non-normal cone metric spaces is followed by some other mathematicians, see [1, 3, 10, 16]. Recently, I. Beg et. al. [2] introduced an appropriate concept of $G$-cone

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metric space which is a generalization of \( G \)-metric space and cone metric space. They also proved some topological properties of these spaces and established some fixed point results for mappings in the setting of \( G \)-cone metric space. Our aim in this study is to obtain sufficient conditions for existence of points of coincidence and common fixed points for three self mappings satisfying certain contractive conditions in \( G \)-cone metric spaces.

2 Preliminaries

We begin by recalling some basic definitions, standard notations and important results for \( G \)-cone metric spaces that will be needed in the sequel.

Let \( E \) be a real Banach Space and \( P \) be a subset of \( E \). Then \( P \) is called a cone if and only if

(i) \( P \) is closed, nonempty and \( P \neq \{0\} \),

(ii) \( a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \) implies \( ax + by \in P \); More generally if \( a, b, c \in \mathbb{R}, a, b, c \geq 0, x, y, z \in P \Rightarrow ax + by + cz \in P \),

(iii) \( P \cap (-P) = \{0\} \).

For a given cone \( P \subseteq E \), we can define a partial ordering \( \leq \) with respect to \( P \) by \( x \leq y \) if and only if \( y - x \in P \). If \( x \leq y \), we write \( y - x = \max\{x, y\} \) and \( x = \min\{x, y\} \). \( x < y \) will stand for \( x \leq y \) and \( x \neq y \), while \( x \ll y \) will stand for \( y - x \in \text{int} P \), where \( \text{int} P \) denotes the interior of \( P \).

A cone \( P \) is called normal if there is a number \( M > 0 \) such that for all \( x, y \in E \),

\[
0 \leq x \leq y \text{ implies } \|x\| \leq M \|y\| .
\]

The least positive number satisfying the above inequality is called the normal constant of \( P \).

Razapour and Hamilbarani [16] proved that there are no normal cones with normal constants \( M < 1 \) and for each \( k > 1 \) there are cones with normal constants \( M > k \).

Definition 2.1. [2], Let \( X \) be a nonempty set. Suppose a mapping \( G : X \times X \times X \to E \) satisfies:

\[
\begin{align*}
(G_1) & G(x, y, z) = 0 \text{ if } x = y = z, \\
(G_2) & 0 < G(x, x, y); \text{ whenever } x \neq y, \text{ for all } x, y \in X, \\
(G_3) & G(x, x, y) \leq G(x, y, z); \text{ whenever } y \neq z, \\
(G_4) & G(x, y, z) = G(x, z, y) = G(y, x, z) = \cdots (\text{Symmetric in all three variables}), \\
(G_5) & G(x, y, z) \leq G(x, a, a) + G(a, y, z) \text{ for all } x, y, z, a \in X.
\end{align*}
\]

Then \( G \) is called a generalized cone metric on \( X \), and \( X \) is called a generalized cone metric space or more specifically a \( G \)-cone metric space.

The concept of a \( G \)-cone metric space is more general than that of a \( G \)-metric space and a cone metric space.

Example 2.1. [2], Let \((X, d)\) be a cone metric space. Define \( G : X \times X \times X \to E \), by

\[
G(x, y, z) = d(x, y) + d(y, z) + d(z, x).
\]

Then \( X \) is a \( G \)-cone metric space.
Definition 2.2. [2], Let $X$ be a $G$-cone metric space and $(x_n)$ be a sequence in $X$. We say that $(x_n)$ is:

(a) Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is $n_0$ such that for all $n, m, l > n_0$, $G(x_n, x_m, x_l) \ll c$.

(b) Convergent sequence if for every $c$ in $E$ with $0 \ll c$, there is $n_0$ such that for all $m, n > n_0$, $G(x_m, x_n, x) \ll c$ for some fixed $x$ in $X$. Here $x$ is called the limit of a sequence $(x_n)$ and is denoted by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

A $G$-cone metric space $X$ is said to be complete if every Cauchy sequence in $X$ is convergent in $X$.

Proposition 2.1. [2], Let $X$ be a $G$-cone metric space then the following are equivalent.

(i) $(x_n)$ converges to $x$.

(ii) $G(x_n, x, x) \to 0$ as $n \to \infty$.

(iii) $G(x_n, x, x) \to 0$, as $n \to \infty$.

(iv) $G(x_m, x_n, x) \to 0$, as $m, n \to \infty$.

Lemma 2.1. [2], Let $X$ be a $G$-cone metric space, $(x_m), (y_n)$, and $(z_l)$ be sequences in $X$ such that $x_m \to x$, $y_n \to y$, and $z_l \to z$, then $G(x_m, y_n, z_l) \to G(x, y, z)$ as $m, n, l \to \infty$.

Lemma 2.2. [2], Let $(x_n)$ be a sequence in a $G$-cone metric space $X$ and $x \in X$. If $(x_n)$ converges to $x$, and $(x_n)$ converges to $y$, then $x = y$.

Lemma 2.3. [2], Let $(x_n)$ be a sequence in a $G$-cone metric space $X$ and $x \in X$. If $(x_n)$ converges to $x$, then $(x_n)$ is a Cauchy sequence.

Lemma 2.4. [2], Let $(x_n)$ be a sequence in a $G$-cone metric space $X$ and if $(x_n)$ is a Cauchy sequence in $X$, then $G(x_m, x_n, x_l) \to 0$ as $m, n, l \to \infty$.

Proposition 2.2. [10], If $E$ is a real Banach space with cone $P$ and if $a \leq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$ then $a = 0$.

Definition 2.3. [15], Let $T$ and $S$ be self mappings of a set $X$. If $w = T(x) = S(x)$ for some $x$ in $X$, then $x$ is called a coincidence point of $T$ and $S$ and $w$ is called a point of coincidence of $T$ and $S$.

Definition 2.4. [15], The mappings $T, S : X \to X$ are weakly compatible, if for every $x \in X$, the following holds:

$$T(S(x)) = S(T(x))$$ whenever $S(x) = T(x)$. 
3 Main Results

In this section we assume that $E$ is a real Banach space, $P$ is a non normal cone in $E$ with $\text{int } P \neq \emptyset$ and $\leq$ is a complete ordering on $E$ with respect to $P$. Throughout the paper we denote by $N$ the set of all positive integers.

We first state a lemma which will play a crucial role in the proof of the main results.

**Lemma 3.1.** [1], Let $X$ be a non empty set and the mappings $S, T, f : X \to X$ have a unique point of coincidence $v$ in $X$. If $(S, f)$ and $(T, f)$ are weakly compatible, then $S, T$ and $f$ have a unique common fixed point.

**Theorem 3.1.** Let $X$ be a $G$-cone metric space and let the mappings $S, T, f : X \to X$ satisfy the following condition:

$$
\max \left\{ \frac{G(S(x), T(y), T(y))}{G(T(x), S(y), S(y))} \right\} \leq a_1 G(f(x), f(y), f(y)) + a_2 \min \left\{ \frac{G(f(x), T(y), T(y)) + G(f(y), S(x), S(x))}{G(f(x), S(y), S(y)) + G(f(y), T(x), T(x))} \right\} + a_3 \min \left\{ \frac{G(f(x), S(x), S(x)) + G(f(y), T(y), T(y))}{G(f(x), T(x), T(x)) + G(f(y), S(y), S(y))} \right\}
$$

(3.1)

for all $x, y \in X$, where $a_1, a_2, a_3 \geq 0$ with $a_1 + 2a_2 + 2a_3 < 1$. If $S(X) \cup T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of $X$, then $S, T$ and $f$ have a unique point of coincidence. Moreover, if $(S, f)$ and $(T, f)$ are weakly compatible, then $S, T$ and $f$ have a unique common fixed point.

**Proof.** Let $x_0 \in X$ be arbitrary. Choose a point $x_1 \in X$ such that $f(x_1) = S(x_0)$, since $S(X) \subseteq f(X)$. Similarly, choose a point $x_2 \in X$ such that $f(x_2) = T(x_1)$. Proceeding in this way, we can define a sequence $(f(x_n))$ by

$$
f(x_n) = \begin{cases} 
S(x_{n-1}), & \text{if } n \text{ is odd} \\
T(x_{n-1}), & \text{if } n \text{ is even}.
\end{cases}
$$

If $n \in N$ is odd, then by using (3.1)
\[ G(f(x_n), f(x_{n+1}), f(x_{n+1})) = G(S(x_{n-1}), T(x_n), T(x_n)) \]
\[ \leq \max \begin{cases} G(S(x_{n-1}), T(x_n), T(x_n)), \\ G(T(x_{n-1}), S(x_n), S(x_n)) \end{cases} \]
\[ \leq a_1 G(f(x_{n-1}), f(x_n)) + a_2 \min \begin{cases} G(f(x_{n-1}), T(x_n), T(x_n)) + G(f(x_n), S(x_{n-1}), S(x_{n-1})), \\ G(f(x_{n-1}), S(x_n), S(x_n)) + G(f(x_n), T(x_{n-1}), T(x_{n-1})) \end{cases} \]
\[ + a_3 \min \begin{cases} G(f(x_{n-1}), S(x_{n-1}), S(x_{n-1})), \\ G(f(x_{n-1}), T(x_{n-1}), T(x_{n-1})) + G(f(x_n), S(x_n), S(x_n)) \end{cases} \].

Thus,
\[ G(f(x_n), f(x_{n+1}), f(x_{n+1})) \leq a_1 G(f(x_{n-1}), f(x_n), f(x_n)) + a_2 \{ G(f(x_{n-1}), T(x_n), T(x_n)) + G(f(x_n), S(x_{n-1}), S(x_{n-1})) \} \]
\[ + a_3 \{ G(f(x_{n-1}), S(x_{n-1}), S(x_{n-1})), + G(f(x_n), T(x_{n-1}), T(x_{n-1})) \} \]
\[ = a_1 G(f(x_{n-1}), f(x_n), f(x_n)) + a_2 \{ G(f(x_{n-1}), f(x_{n+1}), f(x_{n+1})) + G(f(x_n), f(x_n), f(x_n)) \} \]
\[ + a_3 \{ G(f(x_{n-1}), f(x_n), f(x_n)) + G(f(x_n), f(x_{n+1}), f(x_{n+1})) \} \]
\[ \leq a_1 G(f(x_{n-1}), f(x_n), f(x_n)) + a_2 \{ G(f(x_{n-1}), f(x_{n+1}), f(x_{n+1})) + G(f(x_n), f(x_n), f(x_n)) \} \]
\[ + a_3 \{ G(f(x_{n-1}), f(x_n), f(x_n)) + G(f(x_n), f(x_{n+1}), f(x_{n+1})) \} \]

which gives that,
\[ G(f(x_n), f(x_{n+1}), f(x_{n+1})) \leq a_1 + a_2 + a_3 \frac{G(f(x_{n-1}), f(x_n), f(x_n))}{1 - a_2 - a_3} G(f(x_{n-1}), f(x_n), f(x_n)). \]

If \( n \) is even, then by (3.1), we have
\[ G(f(x_n), f(x_{n+1}), f(x_{n+1})) = G(T(x_{n-1}), S(x_n), S(x_n)) \]
\[ \leq \max \begin{cases} G(S(x_{n-1}), T(x_n), T(x_n)), \\ G(T(x_{n-1}), S(x_n), S(x_n)) \end{cases} \]
\[ \leq a_1 G(f(x_{n-1}), f(x_n), f(x_n)) + a_2 \min \begin{cases} G(f(x_{n-1}), T(x_n), T(x_n)) + G(f(x_n), S(x_{n-1}), S(x_{n-1})), \\ G(f(x_{n-1}), S(x_n), S(x_n)) + G(f(x_n), T(x_{n-1}), T(x_{n-1})) \end{cases} \]
\[ + a_3 \min \begin{cases} G(f(x_{n-1}), S(x_{n-1}), S(x_{n-1})), \\ G(f(x_{n-1}), T(x_{n-1}), T(x_{n-1})) + G(f(x_n), S(x_n), S(x_n)) \end{cases} \].
By repeated application of (3.3), we obtain
\[
G(f(x_n), f(x_{n+1}), f(x_{n+1})) \leq a_1 G(f(x_{n-1}), f(x_n), f(x_n)) \\
+ a_2 \{ G(f(x_n-1), S(x_n), S(x_n)) + G(f(x_n), T(x_{n-1}), T(x_{n-1})) \} \\
+ a_3 \{ G(f(x_n-1), T(x_{n-1}), T(x_{n-1})) + G(f(x_n), S(x_n), S(x_n)) \} \\
\leq a_1 G(f(x_{n-1}), f(x_n), f(x_n)) \\
+ a_2 \{ G(f(x_{n-1}), f(x_{n+1}), f(x_{n+1})) + G(f(x_n), f(x_n), f(x_n)) \} \\
+ a_3 \{ G(f(x_{n-1}), f(x_n), f(x_n)) + G(f(x_n), f(x_{n+1}), f(x_{n+1})) \} \\
\leq a_1 G(f(x_{n-1}), f(x_n), f(x_n)) \\
+ a_2 \{ G(f(x_{n-1}), f(x_n), f(x_n)) + G(f(x_n), f(x_{n+1}), f(x_{n+1})) \} \\
+ a_3 \{ G(f(x_{n-1}), f(x_n), f(x_n)) + G(f(x_n), f(x_{n+1}), f(x_{n+1})) \}
\]
which implies that,
\[
G(f(x_n), f(x_{n+1}), f(x_{n+1})) \leq \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3} G(f(x_{n-1}), f(x_n), f(x_n)).
\]

Thus for any positive integer \( n \), it must be the case that,
\[
G(f(x_n), f(x_{n+1}), f(x_{n+1})) \leq \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3} G(f(x_{n-1}), f(x_n), f(x_n)). \tag{3.2}
\]

Let \( r = \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3} \), then \( 0 \leq r < 1 \) since \( a_1, a_2, a_3 \geq 0 \) with \( a_1 + 2a_2 + 2a_3 < 1 \).

Thus, (3.2) becomes
\[
G(f(x_n), f(x_{n+1}), f(x_{n+1})) \leq r G(f(x_{n-1}), f(x_n), f(x_n)). \tag{3.3}
\]

By repeated application of (3.3), we obtain
\[
G(f(x_n), f(x_{n+1}), f(x_{n+1})) \leq r^n G(f(x_0), f(x_1), f(x_1)). \tag{3.4}
\]

Then, for all \( n, m \in \mathbb{N}, \ n < m \), we have by repeated use of (G3) and (3.4) that
\[
G(f(x_n), f(x_m), f(x_m)) \leq G(f(x_n), f(x_{n+1}), f(x_{n+1})) \\
+ G(f(x_{n+1}), f(x_{n+2}), f(x_{n+2})) \\
\vdots \\
+ G(f(x_{m-1}), f(x_m), f(x_m)) \\
\leq \left( r^n + r^{n+1} + \cdots + r^{m-1} \right) G(f(x_0), f(x_1), f(x_1)) \\
\leq \frac{r^n}{1-r} G(f(x_0), f(x_1), f(x_1)).
\]

Let \( 0 \ll c \) be given. Choose \( \delta > 0 \) such that \( c + N_\delta(0) \subseteq \text{int} P \), where \( N_\delta(0) = \{ y \in E : \| y \| < \delta \} \). Also, choose a natural number \( n_0 \) such that
\[
\frac{r^n}{1-r} G(f(x_0), f(x_1), f(x_1)) \in N_\delta(0), \text{ for all } n > n_0.
\]
Then,
\[
\frac{r^n}{1 - r} G(f(x_0), f(x_1), f(x_1)) \ll c, \text{ for all } n > n_0.
\]
Consequently, \( G(f(x_n), f(x_m), f(x_m)) \ll c, \text{ for all } m > n > n_0. \)

Therefore, \( G(f(x_n), f(x_m), f(x_m)) \ll \frac{c}{2}, \text{ for all } m, n > n_0 \) and \( i \geq 1. \) So, in particular
\[
G(f(x_n), f(x_m), f(x_m)) \ll \frac{c}{2}, \text{ for all } m, n > n_0.
\]

For \( n, m, l \in N, \) \((G_5)\) implies that
\[
G(f(x_n), f(x_m), f(x_l)) \leq G(f(x_n), f(x_m), f(x_m)) + G(f(x_l), f(x_m), f(x_m)) \ll \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]
for all \( n, m, l > n_0, \)

This implies that \((f(x_n))\) is a Cauchy sequence in \( f(X). \) By completeness of \( f(X)\), there exist \( u, v \in X \) such that \( f(x_n) \to v = f(u). \)

Further, by \((G_5)\) and \((3.1)\), we have
\[
\begin{align*}
G(f(u), T(u), T(u)) &\leq G(f(u), f(x_{n+1}), f(x_{n+1})) + G(f(x_{n+1}), T(u), T(u)) \\
&= G(f(u), f(x_{n+1}), f(x_{n+1})) + G(S(x_{2n}), T(u), T(u)) \\
&\leq G(f(u), f(x_{n+1}), f(x_{n+1})) + \max \left\{ \begin{array}{l} G(S(x_{2n}), T(u), T(u)), \\ G(T(x_{2n}), S(u), S(u)) \end{array} \right\} \\
&\leq G(f(u), f(x_{n+1}), f(x_{n+1})) + a_1 G(f(x_{2n}), f(u), f(u)) + a_2 \min \left\{ \begin{array}{l} G(f(x_{2n}), T(u), T(u)) + G(f(u), S(x_{2n}), S(x_{2n})), \\ G(f(x_{2n}), S(u), S(u)) + G(f(u), T(x_{2n}), T(x_{2n})) \end{array} \right\} \\
&\quad + a_3 \min \left\{ \begin{array}{l} G(f(x_{2n}), S(x_{2n}), S(x_{2n})) + G(f(u), T(u), T(u)), \\ G(f(x_{2n}), T(x_{2n}), T(x_{2n})) + G(f(u), S(u), S(u)) \end{array} \right\} \\
&\leq G(f(u), f(x_{n+1}), f(x_{n+1})) + a_1 G(f(x_{2n}), f(u), f(u)) + a_2 \{ G(f(x_{2n}), T(u), T(u)) + G(f(u), S(x_{2n}), S(x_{2n})) \} \\
&\quad + a_3 \{ G(f(x_{2n}), S(x_{2n}), S(x_{2n})) + G(f(u), T(u), T(u)) \} \\
&\leq G(f(u), f(x_{n+1}), f(x_{n+1})) + a_1 G(f(x_{2n}), f(u), f(u)) + a_2 \{ G(f(x_{2n}), T(u), T(u)) + G(f(u), f(x_{n+1}), f(x_{n+1})) \} \\
&\quad + a_3 \{ G(f(x_{2n}), f(x_{n+1}), f(x_{n+1})) + G(f(u), T(u), T(u)) \}.
\end{align*}
\]
Taking the limit as \( n \to \infty \), and using Lemma 2.1, we have
\[
G(f(u), T(u), T(u)) \leq (a_2 + a_3) G(f(u), T(u), T(u)). \tag{3.5}
\]
Since \( 0 \leq a_2 + a_3 < 1 \), it follows from (3.5) that,
\[
G(f(u), T(u), T(u)) = 0
\]
which implies that, \( f(u) = T(u) = v \).

Similarly, by using
\[
G(f(u), S(u), S(u)) \leq G(f(u), f(x_{2n+2})f(x_{2n+2})) + G(f(x_{2n+2}), S(u), S(u))
\]
we can show that \( f(u) = S(u) = v \). Thus, \( f(u) = S(u) = T(u) = v \) and so \( v \) becomes a common point of coincidence of \( S, T \) and \( f \).

For uniqueness, let there exists another point \( w \in X \) such that \( f(x) = S(x) = T(x) = w \) for some \( x \in X \).

Then,
\[
G(v, w, w) = G(S(u), T(x), T(x))
\]
\[
\leq \max \left\{ \begin{array}{l}
G(S(u), T(x), T(x)), \\
G(T(u), S(x), S(x))
\end{array} \right\}
\]
\[
\leq a_1 G(f(u), f(x), f(x))
\]
\[
+ a_2 \min \left\{ \begin{array}{l}
G(f(u), T(x), T(x)) + G(f(x), S(u), S(u)), \\
G(f(u), S(x), S(x)) + G(f(x), T(u), T(u))
\end{array} \right\}
\]
\[
+ a_3 \min \left\{ \begin{array}{l}
G(f(u), S(u), S(u)) + G(f(x), T(x), T(x)), \\
G(f(u), T(u), T(u)) + G(f(x), S(x), S(x))
\end{array} \right\}
\]
\[
= a_1 G(v, w, w)
\]
\[
+ a_2 \{ G(v, w, w) + G(w, v, v) \}
\]
\[
+ a_3 \{ G(v, v, v) + G(w, w, w) \}
\]

which gives that,
\[
G(v, w, w) \leq \frac{a_2}{1 - a_1 - a_2} G(w, v, v).
\]

Again by the same argument, we will find that
\[
G(w, v, v) \leq \frac{a_2}{1 - a_1 - a_2} G(v, w, w).
\]

Hence, \( G(v, w, w) \leq \left( \frac{a_2}{1 - a_1 - a_2} \right)^2 G(v, w, w) \). By Proposition 2.2, \( G(v, w, w) = 0 \) which gives that \( v = w \). If \( (S, f) \) and \( (T, f) \) are weakly compatible, then by Lemma 3.1, \( S, T \) and \( f \) have a unique common fixed point in \( X \). \( \square \)
Theorem 3.2. Let $X$ be a $G$-cone metric space and let the mappings $S, T, f : X \rightarrow X$ satisfy the following condition:

$$\max \left\{ G(S(x), T(y), T(y)), \right\} \leq a_1 G(f(x), f(y), f(y))$$

$$\quad + a_2 \min \left\{ G(f(x), f(x), T(y)) + G(f(y), f(y), S(x)), \right\}$$

$$\quad + a_3 \min \left\{ G(f(x), f(x), S(x)) + G(f(y), f(y), T(y)), \right\}$$

for all $x, y \in X$, where $a_1, a_2, a_3 \geq 0$ with $a_1 + 2a_2 + 2a_3 < 1$. If $S(X) \cup T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of $X$, then $S, T$ and $f$ have a unique point of coincidence. Moreover, if $(S, f)$ and $(T, f)$ are weakly compatible, then $S, T$ and $f$ have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary. As in Theorem 3.1, we can define a sequence $(f(x_n))$ by

$$f(x_n) = S(x_{n-1}), \text{ if } n \text{ is odd}$$

$$= T(x_{n-1}), \text{ if } n \text{ is even}.$$ 

Following similar arguments to those given in Theorem 3.1, we have for any positive integer $n$,

$$G(f(x_n), f(x_n), f(x_{n+1})) \leq r^n G(f(x_0), f(x_0), f(x_1)). \quad (3.6)$$

Then, for all $n, m \in \mathbb{N}$, $n < m$, we have by repeated use of $(G_5)$ and (3.6) that

$$G(f(x_m), f(x_n), f(x_n)) \leq G(f(x_m), f(x_{m-1}), f(x_{m-1}))$$

$$+ G(f(x_{m-1}), f(x_{m-2}), f(x_{m-2}))$$

$$\vdots$$

$$+ G(f(x_{n+1}), f(x_n), f(x_n))$$

$$\leq (r^n + r^{n+1} + \cdots + r^{m-1}) G(f(x_0), f(x_0), f(x_1))$$

$$\leq \frac{r^n}{1-r} G(f(x_0), f(x_0), f(x_1)).$$

By an argument similar to that used in the proof of Theorem 3.1, we conclude that $(f(x_n))$ is a Cauchy sequence in $f(X)$. Since $f(X)$ is complete, there exists $u, v \in X$ such that $f(x_n) \rightarrow v = f(u)$.

Now by the same technique as given in Theorem 3.1 and using

$$G(f(u), f(u), S(u)) \leq G(S(u), f(x_{2n+2}), f(x_{2n+2})) + G(f(x_{2n+2}), f(u), f(u))$$
Let
\[
G(f(u), f(u), T(u)) \leq G(T(u), f(x_{2n+1}), f(x_{2n+1})) + G(f(x_{2n+1}), f(u), f(u)),
\]
we obtain \( f(u) = S(u) = T(u) = v \) and \( v \) becomes a common point of coincidence of \( S, T \) and \( f \).
So we get the desired result by Lemma 3.1. \( \square \)

Combining Theorem 3.1 and Theorem 3.2, we state the following Theorem:

**Theorem 3.3.** Let \( X \) be a \( G \)-cone metric space and let the mappings \( S, T, f : X \to X \) satisfy one of the following conditions:

\[
\max \left\{ \frac{G(S(x), T(y), T(y))}{G(T(x), S(y), S(y))} \right\} \leq a_1 G(f(x), f(y), f(y))
\]

\[+a_2 \min \left\{ \frac{G(f(x), T(y), T(y)) + G(f(y), S(x), S(x))}{G(f(x), S(y), S(y)) + G(f(y), T(x), T(x))} \right\}
\]

\[+a_3 \min \left\{ \frac{G(f(x), S(x), S(x)) + G(f(y), T(y), T(y))}{G(f(x), T(x), T(x)) + G(f(y), S(y), S(y))} \right\}
\]

\[\textit{(3.7)}\]

or

\[
\max \left\{ \frac{G(S(x), T(y), T(y))}{G(T(x), S(y), S(y))} \right\} \leq a_1 G(f(x), f(y), f(y))
\]

\[+a_2 \min \left\{ \frac{G(f(x), f(x), T(y)) + G(f(y), f(y), S(x))}{G(f(x), f(x), S(y)) + G(f(y), f(y), T(x))} \right\}
\]

\[+a_3 \min \left\{ \frac{G(f(x), f(x), S(x)) + G(f(y), f(y), T(y))}{G(f(x), f(x), T(x)) + G(f(y), f(y), S(y))} \right\}
\]

\[\textit{(3.8)}\]

for all \( x, y \in X \), where \( a_1, a_2, a_3 \geq 0 \) with \( a_1 + 2a_2 + 2a_3 < 1 \). If \( S(X) \cup T(X) \subseteq f(X) \) and \( f(X) \) is a complete subspace of \( X \), then \( S, T \) and \( f \) have a unique point of coincidence. Moreover, if \( (S, f) \) and \( (T, f) \) are weakly compatible, then \( S, T \) and \( f \) have a unique common fixed point.

**Corollary 3.1.** Let \( X \) be a \( G \)-cone metric space and let the mappings \( T, f : X \to X \) satisfy one of the following conditions:

\[
G(T(x), T(y), T(y)) \leq a_1 G(f(x), f(y), f(y))
\]

\[+a_2 \{ G(f(x), T(y), T(y)) + G(f(y), T(x), T(x)) \}
\]

\[+a_3 \{ G(f(x), T(x), T(x)) + G(f(y), T(y), T(y)) \}
\]
or

\[ G(T(x), T(y), T(y)) \leq a_1 G(f(x), f(y), f(y)) + a_2 \{ G(f(x), f(x), T(y)) + G(f(y), f(y), T(x)) \} + a_3 \{ G(f(x), f(x), T(x)) + G(f(y), f(y), T(y)) \} \]

for all \( x, y \in X \), where \( a_1, a_2, a_3 \geq 0 \) with \( a_1 + 2a_2 + 2a_3 < 1 \). If \( T(X) \subseteq f(X) \) and \( f(X) \) is a complete subspace of \( X \), then \( T \) and \( f \) have a unique point of coincidence. Moreover, if \( T \) and \( f \) are weakly compatible, then \( T \) and \( f \) have a unique common fixed point.

**Proof:** The proof can be obtained from Theorem 3.3 by taking \( S = T \).

**Remark 3.1.** It is worth mentioning that the assumption \( \leq \) is a complete ordering on \( E \) is not required for Corollary 3.1. If \( a_2 = a_3 = 0 \) in Corollary 3.1 we see that it is an extension of the result [15, Theorem 3.2]. Further, taking \( f = I \), the identity map and \( a_1 = a_3 = 0 \) in Corollary 3.1, we obtain the result [2, Theorem 3.4]. So, Corollary 3.1 is both an extension and generalization of some results of [2, 15].

We give an example for Theorem 3.3.

**Example 3.1.** Let \( X = [1, \infty) \), \( E = R \) and \( P = \{ x \in R : x \geq 0 \} \) be a cone in \( E \). Define \( G : X \times X \times X \rightarrow E \) by

\[ G(x, y, z) = |x - y| + |y - z| + |z - x| \text{ for all } x, y, z \in X. \]

Then \( X \) is a complete \( G \)-cone metric space. Define \( T, S, f : X \rightarrow X \) as follows:

\[ T(x) = S(x) = 2x - 1, \]

\[ f(x) = 3x - 2, \]

for all \( x \in X \). Then the following conditions hold trivially:

(i) \( S(X) \cup T(X) \subseteq f(X) \) and \( f(X) \) is complete,

(ii) \( (S, f) \) and \( (T, f) \) are weakly compatible.

Now,

\[
\max \left\{ \begin{array}{c}
G(S(x), T(y), T(y)), \\
G(T(x), S(y), S(y))
\end{array} \right\} = G(T(x), T(y), T(y))
\]

\[
= 2 |T(x) - T(y)|
\]

\[
= 4 |x - y|
\]

\[
= \frac{4}{3} |3x - 3y|
\]

\[
= \frac{4}{3} |f(x) - f(y)|
\]

\[
= \frac{2}{3} G(f(x), f(y), f(y)) \text{ for all } x, y \in X.
\]

If we take \( a_1 = \frac{2}{3} \) and \( a_2 = a_3 = 0 \), then the contractive condition (3.7) or (3.8) of Theorem 3.3 holds good. Thus all the conditions of Theorem 3.3 are satisfied and we see that 1 is the unique common fixed point for \( S, T \) and \( f \).
References


