Numerical solutions of stochastic Lotka-Volterra equations via operational matrices

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Abstract
In this paper, an efficient and convenient method for numerical solutions of stochastic Lotka-Volterra dynamical system is proposed. Here, we consider block pulse functions and their operational matrices of integration. Illustrative example is included to demonstrate the procedure and accuracy of the operational matrices based on block pulse functions.

Keywords: Stochastic operational matrix, Block pulse functions, Stochastic Lotka-Volterra equations, Brownian motion, Itô integral.

1 Introduction
So many phenomena in economics, financial markets, physical systems, biological and medical models give rise to a stochastic differential equations (SDEs) or stochastic Volterra integral equations (SVIEs). Generally, stochastic differential equations combine Gaussian white noise which can be expressed as the derivative of Brownian motion, and since many of them do not have analytic solution, find the numerical solution of them is of special importance, because their solution emulate the behavior of the phenomena.

In this study we consider a stochastic Lotka-Volterra equations and present a method for solving it. The Lotka-Volterra (LV) model has been an active field of research both in the deterministic and stochastic cases since it was originally introduced in the 1920 by Lotka [1] and later applied by Volterra [2] to a predator-prey interaction. This system can model the dynamics of ecological systems with predator-prey interactions, mutualism, disease and competition.

In recent years the numerical study and simulation of stochastic Volterra integral equations have been extensively investigated in the literature [3, 4]. Providing numerical schemes to SDEs have been well developed, however, numerical solution of SVIEs and especially stochastic Lotka-Voltera model is a young field relatively speaking. The authors in [4] solved stochastic Volterra integral equations by stochastic operational matrix based on block pulse functions (BPFs) and in [3] stochastic operational matrix of triangular functions was considered.

This paper is organized as follows. In Section 2, we review block pulse functions and their operational matrices of integration both in the deterministic and stochastic cases. In Section 3, these methods are applied to solve predator-prey system and the numerical results are reported. Finally, Section 4 provides the conclusion.

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2 Block pulse functions (BPFs)

This section contains general information about some typical functions and definitions that are used in the next section. Their properties and operational matrices are reviewed.

BPFs have been variously studied [4] and applied for solving so many equations. The BPFs are defined on the time interval $[0, T)$ by

$$
\psi_i(t) = \begin{cases} 
1 & (i-1)\frac{T}{m} \leq t < i\frac{T}{m}, \\
0 & \text{elsewhere}
\end{cases}
$$

where, $i = 1, \ldots, m$ and for convenient we put $h = \frac{T}{m}$.

The BPFs on $[0, T)$ have some properties as follow:

1. Disjointness: for $i, j = 1, \ldots, m$

$$
\psi_i(t) \psi_j(t) = \delta_{ij} \psi_i(t),
$$

where, $\delta_{ij}$ is Kronecker delta.

2. Orthogonality:

$$
\int_0^T \psi_i(t) \psi_j(t) dt = \delta_{ij} h.
$$

3. Completeness: for every $f \in L^2([0, T))$ when $m$ approaches to infinity, Parseval’s identity holds

$$
\int_0^T f^2(t) dt = \sum_{i=1}^m f_i^2 \| \psi_i(t) \|^2,
$$

where,

$$
f_i = \frac{1}{h} \int_0^T f(t) \psi_i(t) dt.
$$

The set of BPFs can be defined as a $m$ dimension vector $\Psi(t)$

$$
\Psi(t) = [\psi_1(t), \ldots, \psi_m(t)]^T \quad t \in [0, T).
$$

From the above representation and disjointness property, it follows that

$$
\Psi(t) \Psi^T(t) = \begin{pmatrix} 
\psi_1(t) & 0 & 0 & \ldots & 0 \\
0 & \psi_2(t) & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \psi_m(t)
\end{pmatrix}_{m \times m},
$$

where, $F$ is a $m$-dimensional vector and $\tilde{F} = \text{diag}(F)$. The expansion of a function $f(t)$ over $[0, T)$ with respect to $\psi_i(t), i = 1, \ldots, m$ is given by

$$
f(t) \simeq \sum_{i=1}^m f_i \psi_i(t) = F^T \Psi(t) = \Psi^T(t) F.
$$

Now, operational matrix of integration is considered as

$$
\int_0^t \Psi(s) ds \simeq Q \Psi(t),
$$

where, $Q = [q_{ij}]_{m \times m}$ and $q_{ij}$ is defined by (2.5).
where, \( Q \) is operational matrix of integration that is given by

\[
Q = \frac{h}{2} \begin{bmatrix}
1 & 2 & 2 & \ldots & 2 \\
0 & 1 & 2 & \ldots & 2 \\
0 & 0 & 1 & \ldots & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{bmatrix}.
\]  

(2.12)

So,

\[
\int_0^t f(s) ds \simeq \int_0^t F^T \Psi(s) ds \simeq F^T Q \Psi(t).
\]  

(2.13)

Furthermore, the Itô integral of BPFs is defined by

\[
\int_0^t \Psi(t) dB(s) \simeq Q_s \Psi(t),
\]  

(2.14)

where stochastic operational matrix of integration in BPFs domain is

\[
Q_s = \begin{bmatrix}
B(h) & B(h) & B(h) & \ldots & B(h) \\
0 & B(\frac{3h}{2}) - B(h) & B(2h) - B(h) & \ldots & B(2h) - B(h) \\
0 & 0 & B(\frac{5h}{2}) - B(2h) & \ldots & B(3h) - B(2h) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & B(\frac{(2m-1)h}{2}) - B((m-1)h)
\end{bmatrix}_{m \times m}
\]  

(2.15)

So, the Itô integral of every function \( f(t) \) can be approximated by

\[
\int_0^t f(s) dB(s) \simeq \int_0^t F^T \Psi(t) dB(s) \simeq F^T Q_s \Psi(t).
\]  

(2.16)

3 Stochastic Lotka-Volterra model

Lotka-Volterra model also known as the predator-prey equations, in deterministic subclasses, are well-known and have been an active area of research concerning ecological population modeling and economic modeling. These type of equations is so attractive in the terms of economic community, risk management, leadership companies and population dynamics of species competing (conflict). The logistic population model is often represented by [7]:

\[
\frac{dx}{dt} = ax, \quad \frac{dx}{dt} = ax - bx^2.
\]

where \( x \) is the population as a function of \( t \) (time). With logistic dynamics, for two populations, \( x \) and \( y \), the Lotka-Volterra model adds a new phrase to represent the species interactions. So the competitive Lotka-Volterra system takes the following form:

\[
\begin{cases}
\frac{dx(t)}{dt} = x(t)(b_1 - a_{11}x(t) - a_{12}y(t))dt, \\
\frac{dy(t)}{dt} = y(t)(b_2 - a_{21}x(t) - a_{22}y(t))dt.
\end{cases}
\]  

(3.17)

where \( a_{12} \) and \( a_{21} \) are positive constants.

This system can be generalized to any number of species could be in competing against each other, this system of equations with \( n \) interacting components, can be written as

\[
\frac{dx_i(t)}{dt} = x_i(t)(b_i - \sum_{j=1}^{n} a_{ij}x_j(t)), \quad 0 \leq i \leq n.
\]  

(3.18)
However, in this study two species prey-predator interaction is dedicated. Therefore, the system of predator-prey model where the population growth is not limited is given by

\[
\begin{align*}
& dx(t) = x(t)(b_1 - a_{11}x(t))dt, \\
& dy(t) = y(t)(b_2 - a_{21}x(t))dt.
\end{align*}
\] (3.19)

In the recent scientific literature [5, 6, 7], many articles in deterministic case of Lotka-Volterra model, previously discussed, as well as stochastic mode exist. Typically, stochastic phrases are added into the system with assuming that external or internal forces acting on the species have the random nature. The linear set of stochastic Lotka-Volterra equations (linear with the respect to the noise) are as follow:

\[
\begin{align*}
& dx(t) = x(t)(b_1 - a_{11}x(t) - a_{12}y(t))dt + \sigma_1 x(t)dB_1(t), \\
& dy(t) = y(t)(b_2 - a_{21}x(t) - a_{22}y(t))dt + \sigma_2 y(t)dB_2(t).
\end{align*}
\] (3.20)

and the non-linear stochastic Lotka-Volterra model can be achieved as following:

\[
\begin{align*}
& dx(t) = x(t)(b_1 - a_{11}x(t) - a_{12}y(t))dt + \sigma_1 x(t)y(t)dB_1(t), \\
& dy(t) = y(t)(b_2 - a_{21}x(t) - a_{22}y(t))dt + \sigma_2 x(t)y(t)dB_2(t).
\end{align*}
\] (3.21)

### 3.1 Application of operational matrices to solve Lotka-Volterra model

In this section, this method is applied for Eq.(3.20)

\[
\begin{align*}
& x(t) = x_0(t) + \int_0^t b_1 x(s)ds - \int_0^t a_{11} x(s)x(s)ds - \int_0^t a_{12} x(s)\dot{y}(s)ds + \int_0^t \sigma_1 x(s)dB_1(s), \\
& y(t) = y_0(t) + \int_0^t b_2 y(s)ds - \int_0^t a_{21} x(s)x(s)ds - \int_0^t a_{22} y(s)\dot{y}(s)ds + \int_0^t \sigma_2 y(s)dB_2(s),
\end{align*}
\] (3.22)

We approximate function \(x(t), x_0(t), y(t), y_0(t)\) by reported function,

\[
\begin{align*}
& x(t) \simeq \tilde{x}(t) = X^T \Psi(t) = \Psi(t)X, \\
& x_0(t) \simeq X_0^T \Psi(t) = \Psi(t)X_0, \\
& y(t) \simeq \tilde{y}(t) = Y^T \Psi(t) = \Psi(t)Y, \\
& y_0(t) \simeq Y_0^T \Psi(t) = \Psi(t)Y_0,
\end{align*}
\] (3.23-3.26)

where, the vectors \(X, X_0, Y, Y_0\) are BPFs coefficients of \(x, x_0, y, y_0\) respectively.

Substituting (3.23-3.26) into (3.22), we get

\[
\begin{align*}
& X^T \Psi(t) = X_0^T \Psi(s) + \int_0^t b_1 X^T \Psi(s)ds - \int_0^t a_{11} X^T \Psi(s)\dot{X}(s)ds + \int_0^t a_{12} X^T \Psi(s)\dot{Y}(s)ds + \int_0^t \sigma_1 X^T \Psi(s)dB_1(s), \\
& Y^T \Psi(t) = Y_0^T \Phi(t) + \int_0^t b_2 Y^T \Psi(s)ds - \int_0^t a_{21} X^T \Psi(s)\dot{Y}(s)ds + \int_0^t a_{22} Y^T \Psi(s)\dot{Y}(s)ds + \int_0^t \sigma_2 X^T \Psi(s)dB_2(s).
\end{align*}
\] (3.27)

we replace \(X^T \Psi(t)Y^T \Psi(t)\) by \(X^T \tilde{Y} \Psi(t)\).

\[
\begin{align*}
& X^T \Psi(t) = X_0^T \Phi(t) + b_1 X^T Q^T \Psi(t) - a_{11} X^T \dot{X} Q^T \Psi(t) - a_{12} X^T \dot{Y} Q^T \Psi(t) + \sigma_1 X^T Q, \Psi(t), \\
& Y^T \Psi(t) = Y_0^T \Phi(t) + b_2 Y^T Q^T \Psi(t) - a_{21} X^T \dot{Y} Q^T \Psi(t) - a_{22} Y^T \dot{Y} Q^T \Psi(t) + \sigma_2 Y^T Q, \Psi(t).
\end{align*}
\] (3.28)

By replacing \(=\) with \(\simeq\), it gives

\[
\begin{align*}
& X^T = X_0^T + b_1 X^T Q - a_{11} X^T \dot{X} Q + a_{12} X^T \dot{Y} Q + \sigma_1 X^T Q, \\
& Y^T = Y_0^T + b_2 Y^T Q - a_{21} X^T \dot{Y} Q + a_{22} Y^T \dot{Y} Q + \sigma_2 Y^T Q,
\end{align*}
\] (3.29)

after solving nonlinear system (3.29) we find \(X, Y\) and finally \(x(t), y(t)\) of (3.22) are approximated.
3.2 Numerical example

In support of the above statements, initial value \( x(0) = 0.5, y(0) = 1 \), are taken and parameters \( b_1 = 20, b_2 = -30, a_{11} = a_{22} = 0, a_{12} = a_{21} = 25, \sigma_1 = \sigma_2 = 1 \). Then two trajectories of the approximate solution \( x(t) \) and \( y(t) \) are shown in the figure 1.

![Figure 1: Two trajectories of the approximate solution, the blue line represents the results of \( x(t) \) and the red line the results of \( y(t) \) of example by block pulse functions.](image.png)

In the tables, \( n \) is the number of iterations, \( E_x(t) \) and \( E_y(t) \) are means of \( x(t) \) and \( y(t) \) respectively, and \( s_x(t) \) and \( s_y(t) \) are standard deviation of them.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( E_x(t) )</th>
<th>( s_x(t) )</th>
<th>95% Confidence interval for mean of ( x(t) )</th>
</tr>
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<td>.000000000</td>
<td>.500000000 - .500000000</td>
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<td>.10749199</td>
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<tr>
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<td>.37164608</td>
<td>2.2142620 - 2.4802457</td>
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<tr>
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<td>0.4</td>
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<td>.23479506</td>
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4 Conclusion

The approach presented transforms a Lotka-Volterra model into a system of nonlinear algebraic equations. We considered operational matrix of integration both in the deterministic and stochastic cases. The advantage of this method is low cost computing without the need for integration. With these advantages the method is considerably simple and convenient. Performance and efficiency of the proposed method and good degree of accuracy was supported by a numerical example.
Table 2: Mean, standard deviation and confidence interval for mean of $y(t)$ for $m = 16$ with $n = 30$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$E_y(t)$</th>
<th>$s_y(t)$</th>
<th>%95 Confidence interval for mean of $y(t)$</th>
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