Numerical solution of stochastic SIR model via split – step forward Milstein method

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Abstract
The SIR infections disease model is an important biologic model. In this paper, the split-step forward Milstein method, is used for solving numerically stochastic SIR model. The stability of this method is better than the Milsteins methods.

Keywords: Explicit Milstein method, Stochastic SIR model, Stochastic differential equation, Infectious disease.

1 Introduction

W. O. Kermack and A. G. McKendrick introduced a model SIR model. In this model: S(t) shows susceptible to the disease. I(t) represents the number of individuals who have been infected with the disease and R(t) is individuals who have been infected and then removed from the disease. In modeling the spread process of infectious diseases, many classical epidemic models have been proposed and studied, such as SIR, SEIR and SIRS [1-4]. Despite of a century of thorough work, the problem of mathematical description of spread of an epidemic is still an actual question, see for example the review [5], the book [6] and references therein. In recent years, a SIR model with distributed delay has been studied, for example, H.W. Hethcote [7], Hale, S. Verduyn Lunel [8], W. O. Kermack and A. G. Mckendrick [9].

The SIR models is defined as:
\[
\begin{align*}
\dot{S}(t) &= \Lambda - \beta S(t)I(t) - \mu S(t), \\
\dot{I}(t) &= \beta S(t)I(t) + (\mu + \epsilon + \gamma)I(t), \\
\dot{R}(t) &= \gamma I(t) - \mu R(t),
\end{align*}
\] (1.1)

where, the parameter \(\Lambda, \beta, \epsilon, \mu, \gamma\) are positive constants. We can write the stochastic version of (1.1) as:
If a problem is described using the Stratonovich scheme, then the Euler-Heun method has to be used instead of the Euler-Maruyama method that is only valid for Itô SDEs [15, 18].

\[
\begin{align*}
Y_{n+1} &= Y_n - f_n h + \frac{1}{2} [g_n - g(\bar{Y}_n)] \Delta W_n, \\
\bar{Y}_n &= Y_n - g_n \Delta W_n, \\
\Delta W_n &= [W_{t+h} - W_t] \sim \sqrt{h} \cdot N(0, 1).
\end{align*}
\]
Explicit order 1.0 strong Taylor scheme
Milstein method
The Milstein scheme is slightly different whether it is the Itô or Stratonovich representation that is used [11, 13]. It can be proved that Milstein scheme converges strongly with order 1 (and weakly with order 1) to the solution of the SDE. The Milstein scheme represents the order 1.0 strong Taylor scheme.

\[ Y_{n+1} = Y_n + f_n h + g_n \Delta W_n + \frac{1}{2} g_n g_n' (\Delta W_n)^2 - h, \]  
(2.9)

\[ Y_{n+1} = Y_n + f_n h + g_n \Delta W_n + \frac{1}{2} g_n g_n' (\Delta W_n)^2, \]  
(2.10)

\[ \Delta W_n = [W_{t+h} - W_t] \sim \sqrt{h} \ N(0,1). \]  
(2.11)

Where \( g_n' = \frac{dg(Y_n)}{dy_n} \) is the first derivative of \( g_n \). The iterative method defined by (2.9) must be used with Itô SDEs whether (2.10) has to be applied to Stratonovich SDEs. Note that when additive noise is used, i.e. when \( g_n \) is constant and not anymore a function of \( Y_n \) then both Itô and Stratonovich interpretations are equivalent (\( g_n' = 0 \)).

Derivative-free Milstein method
The drawback of the previous method is that it requires the analytic specification of the first derivative of \( g(Y_n) \), analytic expression that can become quickly highly complexe.

The following implementation approximates this derivative thanks to a Runge-Kutta approach [11].

\[ Y_{n+1} = Y_n + f_n h + g_n \Delta W_n + \frac{1}{2 \sqrt{h}} [g(Y_n) - g_n] (\Delta W_n)^2 - h, \]  
(2.12)

\[ Y_{n+1} = Y_n + f_n h + g_n \Delta W_n + \frac{1}{2 \sqrt{h}} [g(Y_n) - g_n] (\Delta W_n)^2, \]  
(2.13)

\[ \bar{Y}_n = Y_n + f_n h + g_n \sqrt{h} \]  
(2.14)

\[ \Delta W_n = [W_{t+h} - W_t] \sim \sqrt{h} \ N(0,1). \]  
(2.15)

where (2.12) and (2.13) must be applied respectively to Itô and Stratonovich SDEs.

Explicit order 1.5 strong Taylor scheme
By adding more terms from a stochastic Taylor expansion than in Milstein scheme, higher strong orders can be obtained. A method to generate a strong order 1.5 method is by introduced Burrage & Platen [14, 15]. For the need of this method, a random variable \( \Delta Z_n \) is introduced.

\[ \Delta Z_n = \int_{T_n}^{T_{n+1}} \int_{T_n}^{T_2} dW_s dS_2 \]  
(2.16)

which is a Gaussian distributed with mean zero, variance \( \frac{1}{3} h^3 \) and correlation

\[ E(\Delta W_n \Delta Z_n) = \frac{1}{2} h^2 \]  
[14, 15].

Stochastic Runge-Kutta
This implementation allows to achieve a 1.5 strong order of converge. This is the highest strong order obtained with a Runge-Kutta approach that keeps a “simple” structure. This implementation makes use of the \( \Delta Z_n \) introduced in (2.16) [14, 15]. Note that this method has been designed for Itô SDEs.

\[ \Delta Y_{n+1} = Y_n + f_n h + g_n \Delta W_n + \frac{1}{2} g_n g_n' (\Delta W_n)^2 - h, \]  
(2.17)

\[ + f_n' g_n \Delta Z_n + \frac{1}{2} f_n f_n' + \frac{1}{2} g_n^2 f_n' \]  
(2.18)

\[ + g_n g_n' \Delta W_n - \Delta Z_n, \]  
(2.19)

\[ + \frac{1}{2} g_n (g_n g_n + (g_n')^2) \frac{1}{3} (\Delta W_n)^2 - h \]  
(2.20)
Convergence
An approximation $Y$ converges with strong order $\gamma > 0$ if there exists a constant $K$ such that [14]

$$E(|X_T - Y_n|) \leq K \cdot h^\gamma$$

(2.21)

for step sizes $h \in (0, 1)$, with $X_T$ being the true solution at time $T$ and $Y_n$ the approximation. The symbol $E$ stands for expectation. It appears that Euler-Maruyama scheme converges only with strong order $\gamma = 0.5$.

Strong approximation is tightly linked to the use of the original increments of the Wiener process [14]. However, in several applications, it is not needed to simulate a pathwise approximation of a Wiener process. For instance, one could be only interested in the moments of the solution of a SDE. A discrete time approximation is tightly linked to the use of the original increments of the Wiener process [14].

Convergence

Maruyama scheme converges only with strong order $\gamma = 0.5$ [16].

3 Split – step forward Milstein method

Let $(\Omega, F, P)$ be probability space with $t \in [t_0, T]$. Consider the following SDE

$$dY(t) = f(Y(t))dt + g(Y(t))d\omega(t),$$

(3.23)

with $Y(t_0) = Y_0$.

Let $gg'$ denote a vector of length $d$ with $ith$ component equal to

$$(g g')_i = \sum_{k=1}^{d} g_k \frac{\partial g_i}{\partial y_k}.$$ 

Suppose that [16, 17]:

A1. For any $x_1, x_2 \in \mathbb{R}^d$, the functions $f$, $g$ and $gg'$ satisfy the Lipschitz condition:

$$|f(x_1) - f(x_2)| \leq L_1 |x_1 - x_2|,$$

(3.24)

$$|g(x_1) - g(x_2)| \leq L_1 |x_1 - x_2|,$$

(3.25)

and

$$|g(x_1)g'(x_1) - g(x_2)g'(x_2)| \leq L_1 |x_1 - x_2|.$$ 

(3.26)

A2. The functions $f$, $g$ and $gg'$ satisfy a linear growth condition; that is,

$$|f(x_1)|^2 \leq C_2 (1 + |x_1|^2),$$

(3.27)

$$|g(x_1)|^2 \leq C_2 (1 + |x_1|^2),$$

(3.28)

$$|g(x_1)g'(x_1)|^2 \leq C_2 (1 + |x_1|^2).$$

(3.29)

Where $C_2$ is a constant.

A3. The process $Y(t)$ is adapted to filtration $\{F_t, t \geq t_0\}$.

Lemma 1. [17] Under the assumptions $A1 - A3$, there exists a unique solution $Y(t)$ to this Equation,

$$dY(t) = f(Y(t))dt + g(Y(t))dW(t), \quad t \in [t_0, T], \quad Y(t_0) = Y_0$$

(3.30)
and
\[
E(\sup_{t \in [0,T]} |Y(t)|^2) < K(1 + E|Y_0|^2),
\]
where \( K \) is a constant.

Consider a uniform step on \([t_0, T]\), \( h = \frac{(T-t_0)}{N} \), \( t_n = t_0 + nh \), where \( n = 0, ..., N \).

The split-step forward Milstein (SSFM) method for SDE (3.30) is [17, 18]:

\[
\begin{align*}
Y_{n1} &= Y_n - \gamma_1 g(Y_n)g'(Y_n)h, \\
Y_{n2} &= Y_{n1} + h\alpha_1 f(Y_{n1}), \\
Y_{n3} &= Y_{n2} + \Delta W_n g(Y_{n2}) + \frac{1}{2}(\Delta W_n)^2 g(Y_{n2})g'(Y_{n2}), \\
Y_{n4} &= Y_{n3} + h\alpha_2 f(Y_{n3}), \\
Y_{n5} &= Y_{n4} - \gamma_2 g(Y_{n4})g'(Y_{n4})h, \\
Y_{n+1} &= Y_{n5} + h\alpha_3 f(Y_{n5}),
\end{align*}
\]

where \( \alpha_1, \alpha_2, \alpha_3, \gamma_1, \gamma_2 \in (-1, 1) \) and \( \alpha_1 + \alpha_2 + \alpha_3 = 1 \), \( \gamma_1 + \gamma_2 = \frac{1}{2} \), and the increments \( \Delta W_n = \mathcal{W}_{n+1} - \mathcal{W}_n \) have \( N(0, h) \) distribution.

Throughout the following analysis, we use \( k_1, k_2, k_3, ..., \) to denote generic constants that do not depend on \( h \).

The SSFM method is consistent with order 2 in the mean and order \( \frac{3}{2} \) in the mean-square sense.

Invoke the Milstein method:
\[
y_{n+1} = y_n + f(y_n)h + g(y_n)\Delta w_n + \frac{1}{2}g(y_n)g'(y_n)[(\Delta w_n)^2 - h],
\]
considered in [13].

4 Application of SSFM method is applied to solve SIR model

Consider the SIR model in Eq. (1.2). Applying split – step forward Milstein discretization in (3.32) for this system of equations we have:
\[
\begin{align*}
S_{n+1} &= S_n + f(S_n)h + g(S_n)\Delta W_n + \frac{1}{2}g(S_n)g'(S_n)[(\Delta W_n)^2 - h], \\
I_{n+1} &= I_n + f(I_n)h + g(I_n)\Delta W_n + \frac{1}{2}g(I_n)g'(I_n)[(\Delta W_n)^2 - h], \\
R_{n+1} &= R_n + f(R_n)h + g(R_n)\Delta W_n + \frac{1}{2}g(R_n)g'(R_n)[(\Delta W_n)^2 - h].
\end{align*}
\]

After solving this nonlinear system (4.33), we find \( S, I, R \) and finally \( S(t), I(t), R(t) \) are approximated.

5 Numerical example

For an example and in order to confirm above results,

Put:
\[
S(0) = 0.7, I(0) = 0.2, R(0) = 0.1 \text{ are chosen and parameters } \Lambda = 0.2, \beta = 0.4, \mu = 0.2, \varepsilon = 0.1, \gamma = 0.2, \sigma_1 = 0.04, \sigma_2 = 0.03, \sigma_3 = 0.02 \text{ and } k = 100.
\]

The numerical results are shown in Table 1, 2 and 3. \( \bar{x}_E \) is the errors mean and and confidence interval of errors in \( k \) iteration.
Table 1: Mean and confidence interval for $S(t)$.

<table>
<thead>
<tr>
<th>$t_i$</th>
<th>$\bar{X}_E$</th>
<th>95% Confidence Interval</th>
<th>Lower bound</th>
<th>Upper bound</th>
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<tbody>
<tr>
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<td>0.700297</td>
<td>0.699762 0.700831</td>
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<tr>
<td>0.1</td>
<td>0.699955</td>
<td>0.699181 0.700729</td>
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<td>0.699213</td>
<td>0.698266 0.700160</td>
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<td>0.3</td>
<td>0.697766</td>
<td>0.696695 0.698837</td>
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<tr>
<td>0.4</td>
<td>0.695897</td>
<td>0.694706 0.697087</td>
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<tr>
<td>0.5</td>
<td>0.693064</td>
<td>0.691784 0.694343</td>
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<tr>
<td>0.6</td>
<td>0.690136</td>
<td>0.688785 0.691487</td>
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<td>0.675082 0.678073</td>
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Table 2: Mean and confidence interval for $I(t)$

<table>
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<tr>
<th>$t_i$</th>
<th>$\bar{X}_E$</th>
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<th>Lower bound</th>
<th>Upper bound</th>
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<tr>
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<tr>
<td>0.1</td>
<td>0.233592</td>
<td>0.233401 0.233783</td>
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<tr>
<td>0.2</td>
<td>0.252335</td>
<td>0.252084 0.252585</td>
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<tr>
<td>0.3</td>
<td>0.272637</td>
<td>0.272330 0.272944</td>
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<tr>
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Table 3: Mean and confidence interval for $R(t)$.

<table>
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<th>$t_i$</th>
<th>$\bar{X}_E$</th>
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<th>Lower bound</th>
<th>Upper bound</th>
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<tr>
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6 Conclusion

This paper suggested a numerical method for solving SIR model by using SSFM method. The significant feature of our method is its better stability [18] as compared to the Milstein method and three stage Milstein methods. From the numerical result (SIR model), it is also clear that the SSFM method is suitable for solving stiff stochastic differential equations. However we believe there is considerable scope for researchers to make important contributions in these areas.
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