Best simultaneous approximation in one-sided $L^1$-NORM

H. Alizadeh Nazarkandi*

Marand Branch, Islamic Azad University, Marand, Iran

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Abstract
Let $K$ be a compact subset of $\mathbb{R}$, $\mu$ be a $\sigma$–finite positive measure, and $s$ an integrable function on $K$. One-sided $L^1$-norm is defined as
\[
\|s\| = \max \left\{ \int_{s \geq 0} |s(x)| d\mu, \int_{s \leq 0} |s(x)| d\mu \right\}.
\]
Best simultaneous approximation of subset $S$ from a subspace $W$ (both of them are subsets of a lattice Banach space as $X$) in this norm is discussed. We give a characterization of best simultaneous approximation for a subset $S$ from subspace $W$ with One-sided $L^1$-norm.

Keywords: Best simultaneous approximation; One-sided $L^1$-norm, Lattice Banach Space.

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1 Introduction
Given a function $s(x)$, let $p(x)$ be another function that approximates $s(x)$. Many norms may be used to measure the size of the error function $e(x) = s(x) - p(x)$. One of them is the $L^1$-norm, which equals the sum of the area above the x-axis and area below the x-axis bounded by $e(x)$. The one-sided $L^1$-norm is an other norm. The norm equals the larger one of the above two areas.

Some results of this paper were obtained earlier in [14] by C. Tang, but there the results were bases on best approximation. We develop most of these works to the best simultaneous approximation case.

We suppose that $K$ be a compact subset of $\mathbb{R}$ and $\mu$ be a $\sigma$–finite positive measure. It can be easily generalized to the where $K$ is a compact subset of a metric space.

Definition 1.1. Let $s$ be an integrable function on $K$. The one-sided $L^1$-norm is defined as follows:
\[
\|s\| = \max \left\{ \int_{s \geq 0} |s(x)| d\mu, \int_{s \leq 0} |s(x)| d\mu \right\}.
\]

C. Yang [14] [ theorem 1.2 ] showed that the one-sided $L^1$-norm defined as above is indeed a norm.

From the definition, the following property is easy to see.
\[
(e1) \quad \|s\| \leq \|s\|_{L^1} \leq 2\|s\|,
\]
Where $\|s\|_{L^1}$ denotes the $L^1$-norm of $s$.

We note that if $H$ be the space of all integrable functions with respect to $\mu$ on $K$ and endowed with $L^1$-norm or one-sided $L^1$-norm then $H$ is clearly a lattice Banach space. In fact, every non-empty upper bounded set admits a supremum. Therefore if $S$ be a bounded subset of $H$ then $\sup S$ exist and $\sup S \in H$ (see [7]).

Let $s \in H$ and $W$ be a subspace of $H$. We recall that function $p \in W$ is called a best approximation to $s$ from $W$ in the one-sided $L^1$-norm, if

$$\|s - p\| \leq \|s - w\|, \text{ for all } w \in W.$$  

The theory of best simultaneous approximation is a generalization of best approximation theory and has been studied by many authors, e.g., [2, 3, 4, 10, 8, 13] and the following definition can be found in these papers.

**Definition 1.2.** Suppose that $X$ is a normed linear space, $W$ a subset of $X$ and $S$ be a bounded set in $X$. We define

$$d(S, W) = \inf_{w \in W} \sup_{s \in S} \|s - w\|.$$  

An element $w_0 \in W$ is called a best simultaneous approximation to $S$ from $W$ whenever $d(S, W) = \sup_{s \in S} \|s - w_0\|$. The set of all best simultaneous approximation to $S$ from $W$ will be denoted by $S_W(S)$. In the case $S = \{x\}, x \in X, S_W(S)$ is the set of all best approximation of $x$ in $W$. It is well-known that $S_W(S)$ is a bounded subset of $X$ and if $W$ is a closed and convex subset of $X$, then $S_W(S)$ is closed and convex.

The results on best simultaneous approximation in general Banach spaces can be found in [5, 11] and the results on best simultaneous approximation in lattice Banach spaces and downward sets can be found in [1, 6].

### 2 Main Results

In the following discussion, $d\mu$ will be omitted in all formulas. We first recall, for the sake of completeness, the following characterization of best simultaneous approximation.

**Theorem 2.1.** [7] Let $W$ be a closed and convex subset of lattice Banach space $X$, $S$ be a bounded set in $X$ with $S \cap W = \emptyset$ and $g_0 \in W$. Then the following assertions are equivalent:

1. $g_0 \in S_W(S)$.
2. There exists $f \in (W - g_0)^0 = \{h \in X^*: h(w - g_0) \leq 0 \text{ } \forall w \in W\}$ such that $\|f\| = 1$ and $f(\sup S - g_0) = \sup_{s \in S} \|s - g_0\|$.

Where $X^*$ is the dual space of $X$.

Let us set, for $w_0 \in H$,

$$Z = Z(w_0) = \{x \in K: w_0(x) = 0\}$$

and

$$N = N(w_0) = K \setminus Z(w_0).$$

**Corollary 2.1.** Let $H$ be the space of all integrable functions with respect to $\mu$ on $K$ and $W$ be a closed and convex subset of $H$, $S$ be a bounded set in $H$ with $S \cap W = \emptyset$. If $w_0 \in W$ is a best simultaneous approximation to $S$ in the $L^1$-norm then we have

$$\int_{N} \text{sgn} (\sup S - w_0)(w - w_0)(x) \leq \int_{Z} |w - w_0| \text{ } \forall w \in W.$$  

**Proof.** By the above theorem there exists a $g \in L^\infty(K)$ for which

1. $\|g\|_{\infty} = 1$;
2. $(g, w - w_0) \leq 0 \text{ } \forall w \in W$.
Thus for small $\varepsilon$, it follows that $w_0 + \varepsilon w \notin S_w (S)$ then the following assertions are true:

1. 

\[
\sup_{s \in S} \int_{s - w_0 > 0} |s - w_0| = \sup_{s \in S} \|s - w_0\| \quad \text{or} \quad \sup_{s \in S} \int_{s - w_0 < 0} |s - w_0| = \sup_{s \in S} \|s - w_0\|.
\]

2. 

\[
\sup_{s \in S} \sup_{s - w_0 > 0} |s - w_0| = \sup_{s \in S} \|s - w_0\| \quad \text{or} \quad \sup_{s \in S} \sup_{s - w_0 < 0} |s - w_0| = \sup_{s \in S} \|s - w_0\|.
\]

**Proof.**

1: First, we prove that

\[
\sup_{s \in S} \int_{s - w_0 > 0} |s - w_0| = \sup_{s \in S} \|s - w_0\| \quad \text{or} \quad \sup_{s \in S} \int_{s - w_0 < 0} |s - w_0| = \sup_{s \in S} \|s - w_0\|.
\]

By definition of the one-sided $L^1$-norm we have

\[
\sup_{s \in S} \int_{s - w_0 > 0} |s - w_0| \leq \sup_{s \in S} \|s - w_0\| \quad \text{and} \quad \sup_{s \in S} \int_{s - w_0 < 0} |s - w_0| \leq \sup_{s \in S} \|s - w_0\|.
\]

Suppose that the state does not hold and

\[
\sup_{s \in S} \int_{s - w_0 > 0} |s - w_0| < \sup_{s \in S} \|s - w_0\| \quad \text{and} \quad \sup_{s \in S} \int_{s - w_0 < 0} |s - w_0| < \sup_{s \in S} \|s - w_0\|.
\]

Then for each $\varepsilon > 0$ and each $s \in S$, \{ $s - w_0 - \varepsilon w > 0$ $|$ $s - w_0 > 0$ $\} \subset \{ s - w_0 > 0 \}$

\[
\int_{s - w_0 - \varepsilon w > 0} |s - w_0 - \varepsilon w| < \int_{s - w_0 > 0} |s - w_0| \leq \sup_{s \in S} \int_{s - w_0 > 0} |s - w_0|.
\]

Since

\[
\lim_{\varepsilon \to 0} \int_{s - w_0 - \varepsilon w < 0} |s - w_0 - \varepsilon w| = \int_{s - w_0 < 0} |s - w_0|.
\]

Thus for small $\varepsilon > 0$,

\[
\int_{s - w_0 - \varepsilon w < 0} |s - w_0 - \varepsilon w| < \int_{s - w_0 < 0} |s - w_0| < \sup_{s \in S} \int_{s - w_0 < 0} |s - w_0| < \sup_{s \in S} \|s - w_0\|.
\]
Therefore for each $s \in S$ and for sufficiently small $\varepsilon > 0$,
\[
||s - w_0 - \varepsilon w|| \leq \sup_{x \in S} ||s - w_0||.
\]
Hence
\[
\sup_{x \in S} ||s - w_0 - \varepsilon w|| \leq \sup_{x \in S} ||s - w_0||.
\]
This shows that $w_0 + \varepsilon w \in S^w(S)$ which is a contradiction.

Now, suppose that the state does not hold and
\[
(i) \quad \sup_{x \in S} \int_{s - w_0 < 0} |s - w_0| < \sup_{x \in S} \int_{s - w_0 > 0} |s - w_0| = \sup_{x \in S} ||s - w_0||
\]
or
\[
(ii) \quad \sup_{x \in S} \int_{s - w_0 > 0} |s - w_0| < \sup_{x \in S} \int_{s - w_0 < 0} |s - w_0| = \sup_{x \in S} ||s - w_0||.
\]
Similar argue leads to contradiction for both of them.

2:
Suppose that the state is not hold. Then
\[
(e3) \quad \int \sup_{s - w_0 > 0} |s - w_0| < \sup_{x \in S} ||s - w_0||
\]
and
\[
(e4) \quad \int \sup_{s - w_0 < 0} |s - w_0| < \sup_{x \in S} ||s - w_0||.
\]
We note that one of the following must be true.
\[
\int \sup_{s - w_0 > 0} |s - w_0| < \int \sup_{s - w_0 < 0} |s - w_0|,
\]
\[
\int \sup_{s - w_0 < 0} |s - w_0| < \int \sup_{s - w_0 > 0} |s - w_0|,
\]
\[
\int \sup_{s - w_0 < 0} |s - w_0| = \int \sup_{s - w_0 > 0} |s - w_0|.
\]
If
\[
\int \sup_{s - w_0 > 0} |s - w_0| < \int \sup_{s - w_0 < 0} |s - w_0|,
\]
then, by definition of one-sided $L^1$-norm, $\sup_{s - w_0} ||s - w_0|| = \int \sup_{s - w_0 < 0} |s - w_0|$. Hence, by (e2) and (e3),
\[
\sup_{x \in S} ||s - w_0|| = \sup_{x \in S} \int_{s - w_0 > 0} |s - w_0| \leq \int \sup_{s - w_0 > 0} |s - w_0| = \sup_{s \in S} ||s - w_0|| < \sup_{x \in S} ||s - w_0||,
\]
which is a contradiction.
If
\[
\int \sup_{s - w_0 < 0} |s - w_0| < \int \sup_{s - w_0 > 0} |s - w_0|,
\]
then, by (e2) and (e4),
\[
\sup_{x \in S} ||s - w_0|| = \int \sup_{s - w_0 > 0} |s - w_0| = \sup_{x \in S} ||s - w_0|| < \sup_{x \in S} ||s - w_0||,
\]
which is a contradiction.
Similarly, we can draw a contradiction if we suppose

\[ \int_{\sup S - w_0 < 0} |\sup S - w_0| = \int_{\sup S - w_0 > 0} |\sup S - w_0|, \]

We now establish a characterization theorem to the one in best approximation in one sided \( L^1 \)-norm.

**Theorem 2.3.** Let \( H \) be the space of all integrable functions with respect to \( \mu \) and \( W \) be a subspace of \( H \) contains a function \( w > 0 \) a.e. \( w_0 \in W \) such that \( w_0 + \epsilon w \notin S_w(S) \). Let \( S \subseteq H \) and \( S \cap W = \emptyset \). \( w_0 \) is a best simultaneous approximation to \( S \) if and only if for each \( w \in W \), either

\[ \int_{\sup S - w_0 < 0} (w - w_0) \leq \int_{\sup S - w_0 = 0} \max \{w_0 - w, 0\} \quad \text{and} \quad \int_{\sup S - w_0 > 0} |\sup S - w_0| = \sup_{s \in S} \|s - w_0\| \]

or

\[ \int_{\sup S - w_0 < 0} (w - w_0) \geq \int_{\sup S - w_0 = 0} \min \{w_0 - w, 0\} \quad \text{and} \quad \int_{\sup S - w_0 < 0} |\sup S - w_0| = \sup_{s \in S} \|s - w_0\|. \]

**Proof.** (\( \Rightarrow \)) Let \( w_0 \) be a best simultaneous approximation to \( S \) from \( W \). Then by Theorem 2.2 we have,

\[ \int_{\sup S - w_0 > 0} |\sup S - w_0| = \sup_{s \in S} \|s - w_0\| \]

or

\[ \int_{\sup S - w_0 < 0} |\sup S - w_0| = \sup_{s \in S} \|s - w_0\|. \]

First let us consider the case of

\[ \int_{\sup S - w_0 > 0} |\sup S - w_0| = \int_{\sup S - w_0 < 0} |\sup S - w_0| = \sup_{s \in S} \|s - w_0\|. \]

Suppose there exist \( w(x) \in W \) such that

\[ -\int_{\sup S - w_0 > 0} (w - w_0) + \int_{\sup S - w_0 = 0} \max \{w_0 - w, 0\} < 0 \]

and

\[ \int_{\sup S - w_0 < 0} (w - w_0) - \int_{\sup S - w_0 = 0} \min \{w_0 - w, 0\} < 0. \]

Then

\[ \int_{\sup S - w_0 = 0} \max \{w_0 - w, 0\} - \int_{\sup S - w_0 = 0} \min \{w - w_0, 0\} < \int_{\sup S - w_0 > 0} (w - w_0) - \int_{\sup S - w_0 < 0} (w - w_0). \]

Thus

\[ \int_{\sup S - w_0 = 0} |w - w_0| < \int_{\sup S - w_0 \neq 0} \text{sgn}(\sup S - w_0)(w - w_0), \]
which contradicts with Corollary 2.1.

Now, let us consider

$$\tag{5} \int_{\sup S - w_0 > 0} |\sup S - w_0| = \sup_{s \in S} |s - w_0| > \int_{\sup S - w_0 < 0} |\sup S - w_0|.$$  

Suppose there exist \(w(x) \in W\) and \(\epsilon < 0\) such that

$$- \int_{\sup S - w_0 > 0} (w - w_0) + \int_{\sup S - w_0 = 0} \max\{w_0 - w, 0\} < c.$$  

Then for \(\epsilon > 0\),

$$\int_{\sup S - w_0 + \epsilon(w_0 - w) > 0} |\sup S - w_0 + \epsilon(w_0 - w)| =$$

$$\int_{\{\sup S - w_0 + \epsilon(w_0 - w) > 0\} \cap \{\sup S - w_0 > 0\}} (\sup S - w_0 + \epsilon(w_0 - w)) +$$

$$\int_{\{\sup S - w_0 + \epsilon(w_0 - w) > 0\} \cap \{\sup S - w_0 \leq 0\}} (\sup S - w_0 + \epsilon(w_0 - w)) =$$

$$\int_{\{\sup S - w_0 > 0\}} (\sup S - w_0 + \epsilon(w_0 - w)) -$$

$$\int_{\{\sup S - w_0 + \epsilon(w_0 - w) \leq 0\} \cap \{\sup S - w_0 > 0\}} (\sup S - w_0 + \epsilon(w_0 - w)) +$$

\[\epsilon \int_{\sup S - w_0 = 0} \max\{w_0 - w, 0\} \leq \int_{\sup S - w_0 > 0} (\sup S - w_0 + \epsilon \int_{\sup S - w_0 > 0} (w_0 - w) -$$

\[\epsilon \int_{\sup S - w_0 + \epsilon(w_0 - w) \leq 0} (w_0 - w) +$$

\[\int_{\{\sup S - w_0 + \epsilon(w_0 - w) \geq 0\} \cap \{\sup S - w_0 < 0\}} \max\{w_0 - w, 0\} \]  

\[\leq \|\sup S - w_0\| + \epsilon c - \int_{\sup S - w_0 + \epsilon(w_0 - w) \leq 0} (\sup S - w_0 + \epsilon \int_{\sup S - w_0 > 0} (w_0 - w) +$$

$$\int_{\sup S - w_0 + \epsilon(w_0 - w) > 0} (w_0 - w).$$

Since

$$\lim_{\epsilon \to 0^+} \mu(\{\sup S - w_0 + \epsilon(w_0 - w) \leq 0\} \cap \{\sup S - w_0 > 0\}) = \lim_{\epsilon \to 0^+} \mu(\{\sup S - w_0 + \epsilon(w_0 - w) > 0\} \cap \{\sup S - w_0 < 0\})$$
\[ e = \lim_{\varepsilon \to 0} \mu \left( \left\{ \sup S - w_0 + \varepsilon (w_0 - w) \right\} \cap \left\{ \sup S - w_0 < 0 \right\} \right) \]

for sufficient small \( \varepsilon > 0 \),

\[ e = \int_{\left\{ \sup S - w_0 + \varepsilon (w_0 - w) \leq 0 \right\} \cap \left\{ \sup S - w_0 > 0 \right\}} (w_0 - w) + \int_{\left\{ \sup S - w_0 + \varepsilon (w_0 - w) > 0 \right\} \cap \left\{ \sup S - w_0 < 0 \right\}} (w_0 - w) < 0. \]

Hence

\[ \int_{\left\{ \sup S - w_0 + \varepsilon (w_0 - w) > 0 \right\}} \left| \sup S - w_0 + \varepsilon (w_0 - w) \right| < \left\| \sup S - w_0 \right\| \leq \sup_{x \in S} \left\| s - w_0 \right\|. \]

On the other hand for small \( \varepsilon > 0 \), and by (e5),

\[ \int_{\left\{ \sup S - w_0 + \varepsilon (w_0 - w) < 0 \right\}} \left| \sup S - w_0 + \varepsilon (w_0 - w) \right| < \left\| \sup S - w_0 \right\| \leq \sup_{x \in S} \left\| s - w_0 \right\|. \]

Thus

\[ \left( e6 \right) \quad \left\| \sup S - w_0 \right\| = \int_{\left\{ \sup S - w_0 > 0 \right\}} \left| \sup S - w_0 \right| = \sup_{x \in S} \left\| s - w_0 \right\| \]

and

\[ \sup_{x \in S} \left\| s - w_0 \right\| = \left\| \sup S - w_0 \right\| = \left\| \sup S - w_0 + \varepsilon (w_0 - w) - \varepsilon (w_0 - w) \right\| \leq \left\| \sup S - w_0 + \varepsilon (w_0 - w) + \varepsilon \right\| w_0 - w \right\|. \]

Therefore, by (e6), we have

\[ \sup_{x \in S} \left\| s - w_0 \right\| + \varepsilon \left\| w_0 - w \right\| \leq \left\| \sup S - w_0 \right\| + \varepsilon \left\| w_0 - w \right\| \leq \left\| \sup S - w_0 + \varepsilon (w_0 - w) \right\| + 2 \varepsilon \left\| w_0 - w \right\| \]

and when the \( \varepsilon \) tends to zero, is a contradiction.

In the case of

\[ \int_{\left\{ \sup S - w_0 > 0 \right\}} \left| \sup S - w_0 \right| < \sup_{x \in S} \left\| s - w_0 \right\| = \int_{\left\{ \sup S - w_0 < 0 \right\}} \left| \sup S - w_0 \right| = \left\| \sup S - w_0 \right\|, \]

the inequality

\[ \int_{\left\{ \sup S - w_0 + \varepsilon (w_0 - w) > 0 \right\}} \left| \sup S - w_0 + \varepsilon (w_0 - w) \right| < \sup_{x \in S} \left\| s - w_0 \right\|, \]

holds for sufficiently small \( \varepsilon > 0 \). This leads to the same contradiction.

(\( \Leftarrow \)) Assume for any \( w(x) \in W \),

\[ \int_{\sup S - w_0 > 0} (w - w_0) \leq \int_{\sup S - w_0 = 0} \max \left\{ w_0 - w, 0 \right\} \]

and
\[
\int_{\sup S - w_0 < 0} \max\{w_0 - w, 0\} \quad \text{and}
\]
\[
\int_{\sup S - w_0 > 0} \max\{w_0 - w, 0\} = \sup_{s \in S} \|s - w_0\|.
\]

Now, if for any \(w(x) \in W\),
\[
\int_{\sup S - w_0 > 0} (w - w_0) \leq \int_{\sup S - w_0 = 0} \max\{w_0 - w, 0\} \quad \text{and}
\]
\[
\int_{\sup S - w_0 > 0} \max\{w_0 - w, 0\} = \sup_{s \in S} ||s - w_0||.
\]

then
\[
\sup_{s \in S} ||s - w|| \geq \int_{\sup S - w_0 > 0} |\sup S - w - w_0| = \int_{\sup S - w_0 > 0} |\sup S - w_0 - (w - w_0)| =
\]
\[
\int_{\{\sup S - w_0 - (w - w_0) > 0\} \cap \{\sup S - w_0 > 0\}} (\sup S - w_0 - (w - w_0)) +
\]
\[
\int_{\{\sup S - w_0 - (w - w_0) > 0\} \cap \{\sup S - w_0 < 0\}} (\sup S - w_0 - (w - w_0)) =
\]
\[
\int_{\{\sup S - w_0 - (w - w_0) > 0\} \cap \{\sup S - w_0 < 0\}} (\sup S - w_0 - (w - w_0)) + \int_{\{\sup S - w_0 < 0\}} \max\{- (w - w_0), 0\} \geq
\]
\[
\int_{\{\sup S - w_0 > 0\}} (\sup S - w_0) - \int_{\{\sup S - w_0 > 0\}} (w - w_0) + \int_{\{\sup S - w_0 = 0\}} \max\{w_0 - w, 0\} \geq
\]
\[
\int_{\{\sup S - w_0 > 0\}} (\sup S - w_0) = \sup_{s \in S} ||s - w_0||.
\]

Similarly, we can draw the same conclusion if
\[
\int_{\sup S - w_0 < 0} (w - w_0) \geq \int_{\sup S - w_0 = 0} \min\{w_0 - w, 0\} \quad \text{and}
\]
\[
\int_{\sup S - w_0 < 0} \min\{w_0 - w, 0\} = \sup_{s \in S} ||s - w_0||.
\]

Thus, \(w_0\) is a best simultaneous approximation.

**Corollary 2.2.** Let \(H\) be the space of all integrable functions with respect to \(\mu\) and \(W\) be a subspace of \(H\), \(S \subseteq H\) and \(S \cap W = \emptyset\) such that \(w_0 < \inf S\) a.e. on \(K\) and \(w_0 + \epsilon w \notin S_W(S)\). Then \(w_0\) is a best simultaneous approximation to \(S\) from \(W\) in the one-sided \(L^1\)-norm if and only if for any \(w \in W\),
\[
0 \leq \int_K w.
\]

(Similarly we can suppose that \(\sup S < w_0\)).
Proof. Let \( w_0 \) be a best simultaneous approximation to \( S \) from \( W \) and \( w_0 < \inf S \) a.e. on \( K \). First, we note that
\[
\sup_{x \in S} \int_{x - w_0 > 0} |s - w_0| = \sup_{x \in S} |s - w_0|.
\]
Suppose that there exists \( w(x) \in W \) and \( c < 0 \) such that \( \int_K w \leq c < 0 \). Then for each \( s \in S \)
\[
\int_K w = \int_{x - w_0 > 0} w + \int_{x - w_0 = 0} w = \int_{x - w_0 > 0} w + \int_{x - w_0 = 0} \max\{w, 0\} \leq c < 0.
\]
The contradiction follows from the above Theorem procedure.
Conversely, assume that for any \( w(x) \in W, 0 \leq \int_K w \). So for each \( s \in S \)
\[
0 \leq \int_K w = \int_{x - w_0 > 0} w + \int_{x - w_0 = 0} \max\{w, 0\}.
\]
The state results from the above Theorem.

Lemma 2.1. Let \( H \) be the space of all integrable functions with respect to \( \mu \) and \( W \) be a subspace of \( H \). Let \( S \subseteq H \) and \( S \cap W = \emptyset \). If for each \( w \in W \),
\[
\int_{\sup S - w_0 > 0} (w - w_0) \leq \int_{\sup S - w_0 = 0} \max\{w_0 - w, 0\}
\]
or
\[
\int_{\sup S - w_0 < 0} (w - w_0) \geq \int_{\sup S - w_0 = 0} \min\{w_0 - w, 0\},
\]
then we have
\[
|\int_{\sup S - w_0 > 0} w| \leq \int_{\sup S - w_0 = 0} |w|.
\]

Proof. Let for each \( w \in W \),
\[
\int_{\sup S - w_0 > 0} (w - w_0) \leq \int_{\sup S - w_0 = 0} \max\{w_0 - w, 0\}.
\]
If we replace \( w \) by \( w + w_0 \) and \( -w + w_0 \) we have, respectively,
\[
\int_{\sup S - w_0 > 0} w \leq \int_{\sup S - w_0 = 0} \max\{-w, 0\} \leq \int_{\sup S - w_0 = 0} |w|,
\]
\[
- \int_{\sup S - w_0 > 0} w \leq \int_{\sup S - w_0 = 0} \max\{w, 0\} \leq \int_{\sup S - w_0 = 0} |w|.
\]
Thus
\[
- \int_{\sup S - w_0 = 0} |w| \leq \int_{\sup S - w_0 > 0} w \leq \int_{\sup S - w_0 = 0} |w|.
\]
That is
\[
|\int_{\sup S - w_0 > 0} w| \leq \int_{\sup S - w_0 = 0} |w|.
\]
Now, let for each \( w \in W \)
\[
\int_{\sup S - w_0 < 0} (w - w_0) \geq \int_{\sup S - w_0 = 0} \min\{w_0 - w, 0\}.
\]
If we replace \( w \) by \( w + w_0 \) and \( -w + w_0 \) we have, respectively,
\[
\int_{\sup S - w_0 > 0} w \geq \int_{\sup S - w_0 = 0} \min\{-w, 0\} = - \int_{\sup S - w_0 = 0} \max\{w, 0\} \geq - \int_{\sup S - w_0 = 0} |w|,
\]
\[
- \int_{\sup S - w_0 > 0} w \geq \int_{\sup S - w_0 = 0} \min\{w, 0\} = - \int_{\sup S - w_0 = 0} \max\{-w, 0\} \geq - \int_{\sup S - w_0 = 0} |w|.
\]
These follow that
\[ -\int_{\sup S-w_0=0}^\infty |w| \leq \int_{\sup S-w_0>0} |w| \leq \int_{\sup S-w_0=0} |w|. \]
That is
\[ |\int_{\sup S-w_0>0} w| \leq \int_{\sup S-w_0=0} |w|. \]

**Theorem 2.4.** Let \( H \) be the space of all integrable functions with respect to \( \mu \) and \( W \) be a subspace of \( H \) contains a function \( w > 0 \) a.e. such that \( w_0 + \varepsilon w \notin S_w(S) \). Let \( S \subseteq H \) and \( S \cap W = \emptyset, w_0 \in W \).

1. Case 1:

\[
\sup_{x \in S} ||s-w_0|| = \int_{\sup S-w_0>0} |\sup S-w_0| > \int_{\sup S-w_0<0} |\sup S-w_0|.
\]

If \( w_0 \in S_w(S) \), then \( w_0 \) is a best simultaneous approximation to \( S \) from \( W \) in the \( L^1 \)-norm on \( \{\sup S-w_0 \geq 0\} \).

2. Case 2:

\[
\sup_{x \in S} ||s-w_0|| = \int_{\sup S-w_0<0} |\sup S-w_0| > \int_{\sup S-w_0>0} |\sup S-w_0|.
\]

If \( w_0 \in S_w(S) \), then \( w_0 \) is a best simultaneous approximation to \( S \) from \( W \) in the \( L^1 \)-norm on \( \{\sup S-w_0 \leq 0\} \).

3. Case 3:

\[
\sup_{x \in S} ||s-w_0|| = \int_{\sup S-w_0<0} |\sup S-w_0| = \int_{\sup S-w_0>0} |\sup S-w_0|.
\]

If \( w_0 \in W \) is a best simultaneous approximation to \( S \) from \( W \) in \( L^1 \)-norm, then \( w_0 \) is a best simultaneous approximation to \( S \) from \( W \) in the one-sided \( L^1 \)-norm.

**Proof.** For the case 1, if \( w_0 \in W \) is a best simultaneous to \( S \) from \( W \) in the one-sided \( L^1 \)-norm, then from Theorem 2.3 and Lemma 2.1, we have
\[ |\int_{\sup S-w_0>0} w| \leq \int_{\sup S-w_0=0} |w|, \text{ for any } w \in W. \]

This shows \( w_0 \) is a best approximation to sup \( S \) from \( W \) in the \( L^1 \)-norm on \( \{\sup S-w_0 \geq 0\} \) by a well-known characterization theorem ([9],Theorem 2.1). Therefore, for each \( w \in W \), we have
\[ \|\sup S-w_0\|_{L^1} \leq \|\sup S-w\|_{L^1} \leq \sup_{x \in S} ||s-w||_{L^1}. \]

Furthermore, based on supposition,
\[
(e7) \quad \|\sup S-w_0\|_{L^1} = \int_{\sup S-w_0 \geq 0} |\sup S-w_0| \\
= \sup_{x \in S} ||s-w_0|| + \int_{\sup S-w_0=0} |\sup S-w_0| = \sup_{x \in S} ||s-w_0||.
\]

On the other hand, for each \( s \in S \),
\[ ||s-w_0||_{L^1} = \int_{\sup S-w_0 \geq 0} |s-w_0| \leq \int_{\sup S-w_0>0} |\sup S-w_0| = \sup_{x \in S} ||s-w_0||, \]
which implies that
\[ \sup_{x \in S} ||s-w_0||_{L^1} \leq \sup_{x \in S} \int_{s-w_0>0} |s-w_0| \leq \sup_{x \in S} ||s-w_0||. \]

This and \( (e7) \) follows that \( w_0 \) is a best simultaneous approximation to \( S \) from \( W \) in the \( L^1 \)-norm on \( \{\sup S-w_0 \geq 0\} \) norm.
The case 2 is similar to the case 1.
For the case 3, we have
\[
\| \sup_{S} S - w_0 \|_{L^1} = \int_{\sup_{S} S - w_0 > 0} |\sup_{S} S - w_0| + \int_{\sup_{S} S - w_0 < 0} |\sup_{S} S - w_0| = 2 \sup_{S \in S} \| s - w_0 \|.
\]
Now by the inequality (e1) we have
\[
2 \sup_{S \in S} \| s - w_0 \| = \| \sup_{S} S - w_0 \|_{L^1} \leq \sup_{S \in S} \| s - w_0 \|_{L^1} \leq \sup_{S \in S} \| s - w \|_{L^1} \leq 2 \sup_{S \in S} \| s - w \|.
\]
That is \( w_0 \) is a best simultaneous approximation to \( S \) from \( W \) in the one-sided \( L^1 \)-norm.

3 Conclusion

When we have an ordered vector space (or lattice space), the results in best approximation may be generalized to the best simultaneous approximation with a little change. Many of these outcomes in two cases have no differences (see [14]). Theorems about the uniqueness of the best approximation in the one-sided \( L^1 \)-norm and best simultaneous approximation do not need to change. Therefore uniqueness in A-space, spline space, weakly chebysheve spaces can be rewritten for best simultaneous approximation with a small change (see [12]). Finally, the additional condition that \( W \) contains a positive almost everywhere function cannot be dropped in the theorems, to see counterclockwise example see [14]).

References


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