A Numerical Approach for Solving Optimal Control Problems Using the Boubaker Polynomials Expansion Scheme

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Abstract
In this paper, we present a computational method for solving optimal control problems and the controlled Duffing oscillator. This method is based on state parametrization. In fact, the state variable is approximated by Boubaker polynomials with unknown coefficients. The equation of motion, performance index and boundary conditions are converted into some algebraic equations. Thus, an optimal control problem converts to a optimization problem, which can then be solved easily. By this method, the numerical value of the performance index is obtained. Also, the control and state variables can be approximated as functions of time. Convergence of the algorithms is proved. Numerical results are given for several test examples to demonstrate the applicability and efficiency of the method.

Keywords: Optimal control problems; Controlled linear and Duffing oscillator; Boubaker polynomials expansion scheme (BPES); optimization problem; Weierstrass approximation theorem.

1 Introduction and Preliminaries
Optimal control problems play an important role in a range of application areas including engineering, economics and finance. Control Theory is a branch of optimization theory concerned with minimizing a cost or maximizing a payout pertaining. An obvious goal is to find an optimal open loop control \( u^*(t) \) or an optimal feedback control \( u^*(t,x) \) that satisfies the dynamical system and optimizes in some sense performance index. There are two general methods for solving optimal control problems. These methods are labeled as direct and indirect methods. An indirect method transforms the problem into another form before solving it and can be grouped into two categories: Bellman’s dynamic programming method and Pontryagin’s Maximum Principle. Bellman pioneered work in dynamic programming which led to sufficient conditions for optimality using the Hamilton-Jacobi-Bellman (HJB) equations. In fact, a necessary condition for an optimal solution of optimal control problems is the HJB equation. It is a second-order partial differential equation which is used for finding a nonlinear optimal feedback control law. Pontryagin’s Maximum Principle is used to find the necessary conditions for the existence of an optimum. This convert the original optimal control problem into a boundary value problem, which can be solved by using well known techniques for differential equations, analytically or numerically (for details see [1, 2, 3, 4, 5, 6]). As analytical solutions of optimal

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Optimal control problems are not always available, therefore, finding a numerical solution for solving optimal control problems is at least the most logical way to treat them and has provided an attractive field for researchers of mathematical sciences. In recent years, different numerical computational methods and efficient algorithms have been used to solve the optimal control problems (for example see [7, 8, 9, 10, 11, 12, 13, 14]).

In direct methods, the optimal solution is obtained by direct minimization of the performance index subject to constraints. In fact, the optimal control problems can be converted into an optimization problem. The direct methods can be employed by using the parameterizations technique which can be applied in three different ways: control parameterizations, control-state parameterizations and state parameterizations [15, 16, 17, 18]. State parameterizations converts the problem to a nonlinear optimization problem and finds unknown polynomial coefficients of degree at most n in the form of \( \sum_{i=0}^{n} a_i t^i \) for optimal solution [19, 20]. The control parameterizations and control-state parameterizations have been used extensively to solve general optimal control problems. Jaddu has presented numerical methods to solve unconstrained and constrained optimal control problems [17] and later, extended his ideas to solve nonlinear optimal control problems with terminal state constraints, control inequality constraints and simple bounds on state variables [18]. In [21, 22], the authors have presented a numerical technique for solving nonlinear constrained optimal control problems. Gindy has presented a numerical solution for solving optimal control problems and the controlled Duffing oscillator [27]. In [25], a numerical technique is shown for solving the controlled Duffing oscillator; in the algorithm the solution is based on state parametrization such that the state variable can be considered as a linear combination of Chebyshev polynomials with unknown coefficients and later, extended state parametrization to solve nonlinear optimal control problems and the controlled Duffing oscillator [27].

This paper is organized into following sections of which this introduction is the first. In Section 2, we introduce mathematical formulation. Section 3 is about Boubaker polynomials. The proposed design approach and its convergence are derived in Section 4. In section 5 we present a numerical example to illustrate the efficiency and reliability of the presented method. Finally, the paper is concluded with conclusion.

2 Problem statement

Optimal control deals with the problem of finding a control law for a given system

\[
x(t) = f(t, x(t), u(t)), \quad t \in I^e,
\]

where, \( f \) is a real-valued continuously differentiable function, \( f : I \times E \times U \rightarrow \mathbb{R}^n \). Also \( I = [t_0, t_1] \) for the time interval, \( u(t) : I \rightarrow \mathbb{R}^m \) for the control and \( x(t) : I \rightarrow \mathbb{R}^n \) for the state variable is used. As the control function is changed, the solution to the differential equation will be changed. The subject is to find a piecewise continuous control \( u^* \) and the associated state variable \( x^*(t) \) that optimizes in some sense the performance index

\[
J(t_0, x_0; u) = \int_{t_0}^{t_1} \mathcal{L}(t, x(t), u(t)) dt,
\]

subject to (2.1) with boundary conditions

\[
x(t_0) = x_0 \quad \text{and} \quad x(t_1) = x_1,
\]

where, \( x_0 \) and \( x_1 \) are initial and final state in \( \mathbb{R}^n \); respectively, that may be fixed or free. Control \( u^* \) is called an optimal control and state variable \( x^* \) an optimal trajectory. Also, \( L : I \times E \times U \rightarrow \mathbb{R} \) is assumed to be a continuously differentiable function in all three arguments, the optimization problem with performance index as in equation (2.2) is called a Lagrange problem. There are two other equivalent optimization problems, which are called Bolza and Mayer problems [2]. Particularly in optimal control problems \( L \) can be an energy or fuel function as below [28]:

\[
\begin{align*}
L(t, x(t), u(t)) &= \frac{1}{2}(x^2(t) + u^2(t)), \\
L(t, x(t), u(t)) &= |x(t)| + |u(t)|.
\end{align*}
\]
Generally, $J$ may be a multi purpose or multi objective functional; for example, in minimization of fuel dissipation or maximization of benefit.

Example 2.1. Linear quadratic control problem

A large number of design problems in engineering is an optimization problem. This problem is called the linear regulator problem. Let $A(t)$, $M(t)$ and $D$ be $n \times n$ matrices and $B(t)$, $n \times m$ and $N(t)$, $m \times m$, matrices of continuous functions. Let $u(t)$ be defined on a fixed interval $[t_0,t_1]$, which is an $m$–dimensional piecewise continuous vector function. The state vector $x(t) \in \mathbb{R}^n$ is the corresponding solution of initial value problem

$$
\dot{x} = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0. \tag{2.4}
$$

The optimal control problem is to find an optimal control $u(t)$ which minimizes the performance index,

$$
J = x(t_1)'Dx(t_1) + \int_{t_0}^{t_1} \left( x(t)'M(t)x(t) + u(t)'N(t)u(t) \right) dt. \tag{2.5}
$$

Here $M(t)$, $N(t)$ and $D$ are symmetric with $M(t)$ and $D$ non negative definite and $N(t)$ positive definite matrices. Let $x = (x_1,x_2,\cdots,x_n)'$ then the Hamilton–Jacobi–Bellman equation with the final condition will be

$$
(HJB) \quad \begin{cases} V_t + \min_{u\in U} \left( V_x(Ax + Bu) + (x'Mx + u'Nu) \right) = 0, \\ V(t_1,x) = x'Dx. \end{cases}
$$

In the case of linear quadratic optimal control problem (2.4)-(2.5), if the value $V(t,x) = x'K(t)x$ is substituted in the HJB equation, where $K(t)$ is a $C^1$ symmetric matrix with $K(t_1) = D$, then HJB equation leads to a control law of the form

$$
u(t) = -N^{-1}(t)B(t)'K(t)x(t).$$

Here $K(t)$ satisfies the matrix Riccati equation [17]

$$\begin{cases} \dot{K}(t) = -A(t)'K(t) - K(t)A(t) + K(t)B(t)N^{-1}(t)B(t)'K(t) - M(t), \\ K(t_1) = D. \end{cases}$$

Example 2.2. The Controlled Linear Oscillator

We will consider the optimal control of a linear oscillator governed by the differential equation

$$
u(t) = x(t) + \omega^2x(t), \quad t \in [-T,0], \tag{2.6}
$$

in which $T$ is specified. Equation (2.6) is equivalent to the dynamic state equations

$$
\begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= -\omega^2x_1(t) + u(t),
\end{align*}
$$

with the boundary conditions

$$
x_1(-T) = x_0, \quad x_2(-T) = x_0, \\
x_1(0) = 0, \quad x_2(0) = 0. \tag{2.7}
$$

It is desired to control the state of this plant such that the performance index

$$
J = \frac{1}{2} \int_{-T}^{0} u^2(t) dt, \tag{2.8}
$$

Their coefficients could be defined through a recursive formula where:

\[ x_2(t) = \frac{1}{2\omega} [A(\omega t \sin \omega t + \omega t \cos \omega t) + B t \sin \omega t], \]

\[ u(t) = A \cos \omega t + B \sin \omega t, \]

\[ J = \frac{1}{8\omega^2} [2\omega T (A^2 + B^2) + (A^2 - B^2) \sin 2\omega T - 4AB \sin^2 \omega T], \]

where

\[ A = \frac{2\omega [x_0 \omega^2 T \sin \omega T - x_0 (\omega T \cos \omega T - \sin \omega T)]}{\omega^2 T^2 - \sin^2 \omega T}, \]

\[ B = \frac{2\omega^2 [x_0 T \sin \omega T + x_0 (\omega T \cos \omega T + \sin \omega T)]}{\omega^2 T^2 - \sin^2 \omega T}. \]

The Controlled Duffing Oscillator

Controlled Duffing oscillator described by the nonlinear differential equation

\[ u(t) = \ddot{x}(t) + \omega^2 x(t) + \varepsilon \dot{x}^3(t), \quad t \in [-T, 0], \]

Subject to the boundary conditions and with the performance index pointed out as in the previously linear case. The exact solution in this case is not known.

3 The Boubaker polynomials

In this section, Boubaker polynomials, which are used in the next sections, are reviewed briefly. The Boubaker polynomials were established for the first by Boubaker et al. as a guide for solving heat equation inside a physical model. In fact, in a calculation step during resolution process, an intermediate calculus sequence raised an interesting recursive formula leading to a class of polynomial functions that performs difference with common classes (for details see [29, 30, 31, 32, 33, 34]).

**Definition 3.1.** The first monomial definition of the Boubaker polynomials is introduced by:

\[ B_n(X) = \sum_{p=0}^{\zeta(n)} \left[ \frac{(n-4p)}{(n-p)} C_{n-p} \right] (-1)^p X^{n-2p}, \]

where

\[ \zeta(n) = \left[ \frac{n}{2} \right] = \frac{2n + ((-1)^n - 1)}{4}. \]

Their coefficients could be defined through a recursive formula

\[
\begin{cases}
B_n(t) = \sum_{j=0}^{\zeta(n)} [b_{n,j} t^{n-2j}], \\
b_{n,0} = 1, \\
b_{n,1} = -(n-4), \\
b_{n,j+1} = \frac{(n-2j)(n-2j-1)}{(j+1)(n-j-1)} b_{n,j} - b_{n,j}, \\
b_{n,\zeta(n)} = \left\{ \begin{array}{ll}
-\frac{1}{2} & \text{if } n \text{ even}, \\
(\pm 1)^{\frac{n+1}{4}} (n-2) & \text{if } n \text{ odd},
\end{array} \right.
\end{cases}
\]
Remark 3.1. A recursive relation which yields the Boubaker polynomials is:

\[
\begin{align*}
B_0(t) &= 1, \\
B_1(t) &= t, \\
B_2(t) &= t^2 + 2, \\
B_m(t) &= tB_{m-1}(t) - B_{m-2}(t), \quad \text{for } m > 2,
\end{align*}
\]

The ordinary generating function \( f_B(t,x) \) of the Boubaker polynomials is:

\[
f_B(t,x) = \frac{1 + 3x^2}{1 + x(x-t)}.
\]

The characteristic differential equation of the Boubaker polynomials is:

\[
A_n\dddot{x} + B_n\ddot{x} - C_n x = 0.
\]

where

\[
\begin{align*}
A_n &= (x^2 - 1)(3nx^2 + n - 2), \\
B_n &= 3x(nx^2 + 3n - 2), \\
C_n &= -n(3n^2x^2 + n^2 - 6n + 8).
\end{align*}
\]

Lemma 3.1. Some arithmetical or integral properties of Boubaker polynomials are as follow:

\[
\begin{align*}
B_n(0) &= 2\cos\left(\frac{n+2}{2}\pi\right), \quad n \geq 1 \\
B_n(-t) &= (-1)^nB_n(t).
\end{align*}
\]

4 The proposed design approach

In this section, a new parameterizations using Boubaker polynomials, to derive a robust method for solving optimal control problems numerically is introduced. In fact, we can accurately represent state and control functions with only a few parameters. First, from equation (2.1), the expression for \( u(t) \) as a function of \( t, x(t) \) and \( \dot{x}(t) \) is determined, i.e. [10]

\[
u(t) = \phi(t, x(t), \dot{x}(t)), \quad (4.9)
\]

Let \( Q \subset C^1([0,1]) \) be set of all functions satisfying initial conditions (2.3). Substituting (4.9) into (2.2), shows that performance index (2.2) can be explained as a function of \( x \). Then, the optimal control problem (2.1)-(2.3) may be considered as minimization of \( J \) on the set \( Q \). The state parametrization can be employed using different basis functions [16]. In this work, Boubaker polynomial will be applied to introduce a new algorithm for solving optimal control problems numerically. Let \( Q_n \subset Q \) be the class of combinations of Boubaker polynomials of degrees up to \( n \), and consider the minimization of \( J \) on \( Q_n \) with \( \{a_k\}_{k=0}^n \) as unknowns. In fact, state variable is approximated as follow:

\[
x_n(t) = \sum_{k=0}^n a_kB_k(t), \quad n = 1,2,3,\ldots \quad (4.10)
\]

The control variables are determined from the system state equations (4.9) as a function of the unknown parameters of the state variables

\[
u_n(t) = \phi(t, \sum_{k=0}^n a_kB_k(t), \sum_{k=0}^n a_k\dot{B}_k(t)). \quad (4.11)
\]

By substituting these approximation of the state variables (4.10) and control variables (4.11) into the performance index (2.2) yield:

\[
J(a_0,a_1,\ldots,a_n) = \int_{t_0}^{t_1} L(t, \sum_{k=0}^n a_kB_k(t), \phi(t, \sum_{k=0}^n a_kB_k(t), \sum_{k=0}^n a_k\dot{B}_k(t)))dt. \quad (4.12)
\]
Thus, the problem can be converted into a quadratic function of the unknown parameters \( a_i \). The initial condition is replaced by equality constraint as follow:

\[
x_n(t_0) = \sum_{k=0}^{n} a_k B_k(t) \bigg|_{t=t_0} = x_0,
\]

\[
x_n(t_1) = \sum_{k=0}^{n} a_k B_k(t) \bigg|_{t=t_1} = x_1.
\]

The new problem can be stated as:

\[
\min_{a \in \mathbb{R}^{n+1}} \{a^H a\},
\]

subject to constraints (4.13) due to the initial and final conditions, which are linear constrains as:

\[
P a = b.
\]

In fact, this is an optimization problem in \((n + 1)\)-dimensional space and \( J(x_n) \) may be considered as \( \hat{J}(a') = \hat{J}(a_0, a_1, \ldots, a_n) \), which \( \hat{J} \) is approximate value of \( J \). The optimal value of the vector \( a^* \) can be obtained from the standard quadratic programming method.

### 4.1 An efficient algorithm

The above result is summarized in the following algorithm. The main idea of this algorithm is to transform the optimal control problems (2.1)-(2.3) into a optimization problem (4.14)-(4.15) and then solve this optimization problem.

**Algorithm.**

**Input:** Optimal control problem (2.1)-(2.3).

**Output:** The approximate optimal trajectory, approximate optimal control and approximate performance index \( J \).

**Step 1.** Approximate the state variable by \( n^{th} \) Boubaker series from equation (4.10).

**Step 2.** Find the control variable as a function of the approximated state variable from equation (4.11).

**Step 3.** Find an expression of \( \hat{J} \) from equation (4.12) and find the matrix \( H \).

**Step 4.** Determine the set of equality constraints, due to the initial and final conditions and find the matrix \( P \).

**Step 5.** Determine the optimal parameters \( a^* \) by solving optimization problem (4.14)-(4.15) and substitute these parameters into equations (4.10), (4.11) and (4.12) to find the approximate optimal trajectory, approximate optimal control and approximate performance index \( J \), respectively.

### 4.2 A case study

The next example clarifies the presented concepts:

Find \( u^*(t) \) that minimizes [17]

\[
J = \int_{0}^{1} (x_1^2 + x_2^2 + 0.0005u^2(t))dt, \quad 0 \leq t \leq 1,
\]

subject to

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -x_2 + u,
\end{align*}
\]

with initial conditions

\[
x_1(0) = 0, \quad x_2(0) = -1.
\]
The first step in solving this problem by the proposed method is by approximating \( x_1(t) \) by 5\(^{th}\) order Boubaker series of unknown parameters, we get

\[
x_1(t) = \sum_{k=0}^{5} a_k B_k(t) = a_0 t^5 + a_1 t^4 + (a_3 - a_5) t^3 + a_2 t^2 + (a_1 + a_3 - 3 a_5) t + a_0 + 2 a_2 - 2 a_4. \tag{4.19}
\]

Then, \( \dot{x}_1(t) \) is calculated and \( x_2(t) \) can be determined,

\[
x_2(t) = 5 a_0 t^4 + 4 a_1 t^3 + (3 a_3 - 3 a_5) t^2 + 2 a_2 t + a_1 + a_3 - 3 a_5, \tag{4.20}
\]

and then the control variable are obtained from the state equation (4.17), as follows:

\[
u(t) = 5 a_0 t^4 + (4 a_1 + 20 a_3) t^3 + (3 a_3 + 12 a_4 - 3 a_5) t^2 + (2 a_2 + 6 a_3 - 6 a_5) t + a_1 + 2 a_2 + a_3 - 3 a_5, \tag{4.21}
\]

By substituting (4.19)-(4.21) into (4.16), the following expression for \( \hat{J} \) can be obtained

\[
\hat{J} = -a_0^2 + 2 a_0 a_1 + \frac{13}{3} a_0 a_2 + \frac{3}{2} a_0 a_3 - \frac{18}{5} a_0 a_4 - \frac{19}{6} a_0 a_5 + \frac{803}{600} a_1^2 + \frac{453}{60} a_1 a_2 + \frac{307}{60} a_1 a_3 + \frac{23}{60} a_1 a_4
\]

\[-\frac{5687}{700} a_1 a_5 + \frac{1037}{150} a_2^2 + \frac{5399}{600} a_2 a_3 - \frac{12788}{2625} a_2 a_4 - \frac{1631}{120} a_2 a_5 + \frac{122539}{21000} a_2 a_6 + \frac{593}{150} a_3 a_4 - \frac{461117}{31500} a_3 a_5
\]

\[+ \frac{45929}{7875} a_3^2 + \frac{263}{600} a_3 a_5 + \frac{8841737}{693000} a_5^2, \tag{4.22}
\]

which convert to

\[
J = \begin{bmatrix}
1 & \frac{1}{2} & \frac{7}{3} & \frac{3}{4} & -\frac{9}{5} & -\frac{19}{12}
\end{bmatrix} \begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5
\end{bmatrix} = \begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5
\end{bmatrix}. \tag{4.23}
\]

From initial conditions (4.18), another equations representing the initial states are obtained as follow:

\[
a_0 + 2 a_2 - 2 a_4 = 0,
\]

\[
a_1 + a_3 - 3 a_5 = -1, \tag{4.24}
\]

that means:

\[
\begin{bmatrix}
1 & 0 & 2 & 0 & -2 & 0 \\
0 & 1 & 0 & 1 & 0 & -3
\end{bmatrix} \begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5
\end{bmatrix} = \begin{bmatrix}
0 \\
-1
\end{bmatrix}. \tag{4.25}
\]

The dynamic optimal control problem is approximated by a quadratic programming problem. The new problem is to minimize (4.23) subject to the equality constraint (4.25). The optimal value of the vector \( a^* \) can be obtained from the
standard quadratic programming method as:

\[
\begin{bmatrix}
  a_0 \\
  a_1 \\
  a_2 \\
  a_3 \\
  a_4 \\
  a_5 
\end{bmatrix}
= \begin{bmatrix}
  100962514997240 \\
  1281600273247307 \\
  61331171268920 \\
  563785654553864 \\
  111812428767540 \\
  103630641719934
\end{bmatrix}
\]

(4.26)

Now, we calculate state variables \( x_1(t) \) and \( x_2(t) \) approximately as:

\[
x_1(t) = -t + \frac{61331171268920}{12724695632333} t^2 - \frac{866845180763766}{89072869426331} t^3 + \frac{111812428767540}{12724695632333} t^4 - \frac{1303630641719934}{445364347131655} t^5,
\]

and

\[
x_2(t) = -1 + \frac{122662342537840}{12724695632333} t - \frac{2600535542291298}{89072869426331} t^2 + \frac{447249715070160}{12724695632333} t^3 - \frac{1303630641719934}{89072869426331} t^4,
\]

and approximated control \( u(t) \) as:

\[
u(t) = \frac{109937646905507}{12724695632333} t - \frac{4342434686817716}{89072869426331} t^2 + \frac{6791708474182062}{89072869426331} t^3 - \frac{2083774561388616}{89072869426331} t^4.
\]

Also, by substituting optimal parameters (4.26) into (4.23) the approximate optimal value can be obtained. For the example, the optimal value is obtained \( J = 0.0759522 \). This particular case is also solved by approximating \( x_1(t) \) into 9th order Boubaker series of unknown parameters. The optimal value is obtain to be 0.0693689 which is very close to both the exact value 0.06936094 and the result obtained in [17] using 9th order Chebyshev series which is 0.0693689

### 4.3 Convergence analysis

The convergence analysis of the proposed method is based on Weierstrass approximation theorem (1885).

**Theorem 4.1.** Let \( f \in C([a,b], \mathbb{R}) \). Then there is a sequence of polynomials \( P_n(x) \) that converges uniformly to \( f(x) \) on \([a,b]\).

**Proof.**

See [35]

**Lemma 4.1.** If \( \alpha_n = \inf_{P_n} J \) for \( n \in \mathbb{N} \), where \( P_n \) be a subset of \( Q \), consisting of all polynomials of degree at most \( n \). Then \( \lim_{n \to \infty} \alpha_n = \alpha \) where \( \alpha = \inf_{Q} J \).

**Proof.**

See [26]

The next theorem guarantees the convergence of the presented method to obtain the optimal performance index \( J(\cdot) \).

**Theorem 4.2.** If \( J \) has continuous first derivatives and for \( n \in \mathbb{N} \), \( \gamma_n = \inf_{Q_n} J \). Then \( \lim_{n \to \infty} \gamma_n = \gamma \) where \( \gamma = \inf_{Q} J \).
Proof.
If we define \( \gamma_n = \min_{a_n \in \mathbb{R}^{n+1}} J(a_n) \), then:

\[
\gamma_n = J(a^*_n), \quad a^*_n \in \text{Argmin}\{J(a_n) : a_n \in \mathbb{R}^{n+1}\}.
\]

Now let:

\[
x^*_n \in \text{Argmin}\{J(x(t)) : x(t) \in Q_n\}.
\]

Then

\[
J(x^*_n(t)) = \min_{x(t) \in Q_n} J(x(t)),
\]

in which \( Q_n \) is a class of combinations of Boubaker polynomials in \( t \) of degree \( n \), so \( \gamma_n = J(x^*_n(t)) \). Furthermore, according to \( Q_n \subset Q_{n+1} \), we have:

\[
\min_{x(t) \in Q_{n+1}} J(x(t)) \leq \min_{x(t) \in Q_n} J(x(t)).
\]

Thus, we will have \( \gamma_{n+1} \leq \gamma_n \) which means \( \gamma_n \) is a non increasing sequence. Now, according to Lemma 4.1, the proof is complete, that is:

\[
\lim_{n \to \infty} \gamma_n = \min_{x(t) \in Q} J(x(t)).
\]

Note that, this theorem is proved when \( Q_n \) is a class of combinations of Chebyshev polynomials [26].

5 Numerical examples

To illustrate the efficiency of the presented method, we consider the following examples. All problems considered have continuous optimal controls and can be solved analytically. This allows verification and validation of the method by comparing with the results of exact solutions. Note that, our method is based on state parameterization, so we have compared it with the method given in [20], [26] and [27]. Furthermore, comparison between the exact and the approximate trajectory of \( x(t) \), of control \( u(t) \) and of performance index \( J \) are also presented (see tables 2 and 6 also table 4).

Example 5.1. ([2, 7, 8, 23, 26])
The object is to find the optimal control which minimizes

\[
J = \frac{1}{2} \int_0^1 (u^2(t) + x^2(t)) dt, \quad 0 \leq t \leq 1,
\]

when

\[
\dot{x}(t) = -x(t) + u(t), \quad x(0) = 1.
\]

We can obtain the analytical solution by the use of Pontryagin’s maximum principle which is [26]:

\[
x(t) = Ae^{\sqrt{2}t} + (1 - A)e^{-\sqrt{2}t},
\]

\[
u(t) = A(\sqrt{2} + 1)e^{\sqrt{2}t} - (1 - A)(\sqrt{2} - 1)e^{-\sqrt{2}t},
\]

and

\[
J = \frac{e^{-2\sqrt{2}}}{2} \left( (\sqrt{2} + 1)(e^{4\sqrt{2}} - 1)A^2 + (\sqrt{2} - 1)(e^{2\sqrt{2}} - 1)(1 - A)^2 \right).
\]
where $A = \frac{2\sqrt{2} - 3}{\sqrt{2} + 2\sqrt{2} - 3}$. By approximating $x(t)$ by second order Boubaker series of unknown parameters, we get

$$x(t) = \sum_{k=0}^{2} a_k B_k(t), \quad (5.29)$$

and then the control variable are obtained from the state equation (5.28), as follows:

$$u(t) = +a_2 t^2 + (a_1 + 2a_2)t + a_0 + a_1 + 3a_2. \quad (5.30)$$

By substituting (5.29) and (5.30) into (5.27), the following expression for $\hat{J}$ can be obtained

$$\hat{J} = \begin{bmatrix} a_0 & a_1 & a_2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 17/6 \\ 1 & 4 & 13/4 \\ 17/6 & 13/4 & 87/10 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}. \quad (5.31)$$

From initial condition (5.28), another equation representing the initial states are obtained

$$a_0 + 2a_2 = 1. \quad (5.32)$$

The dynamic optimal control problem is approximated by a quadratic programming problem. The new problem is to minimize (5.31) subject to the equality constraint (5.32). The optimal value of the vector $a^*$ can be obtained from the standard quadratic programming method. By substituting these optimal parameters into (5.31), the approximate optimal value can be calculated. The optimal cost functional $J$, obtained by the presented method is shown for different $n$ in Table 1.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$J$</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.1929316056</td>
<td>$2.2\times10^{-5}$</td>
</tr>
<tr>
<td>4</td>
<td>0.1929094450</td>
<td>$1.7\times10^{-7}$</td>
</tr>
<tr>
<td>5</td>
<td>0.1929092990</td>
<td>$8.6\times10^{-10}$</td>
</tr>
</tbody>
</table>

The exact solution for the performance index is $J = 0.1929092978$. The optimal cost functional, $J$, obtained by the presented method for $n = 4$ is a good approximation. This leads to state and control variables approximately as:

$$x(t) = 1.0 - 1.38t + 0.982t^2 - 0.4081t^3 + 0.0871t^4,$$

and

$$u(t) = -0.384 + 0.579t - 0.227t^2 - 0.0542t^3 + 0.0871t^4,$$

The obtained solution and the analytical solution are plotted in Figure 1.
Figure 1: Solution of Example 5.1. The approximate solution for \( n = 4 \) is compared with the actual analytical solution.

Note that, the previous problem is also solved by expanding \( x(t) \) into 10th order Boubaker series, and the optimal value is obtained 0.19290992981 which is very close to the exact value \( J \) and the result obtained in [20] and [26], three iterations of their algorithms are 0.193828723 and 0.192909776, respectively. The maximum absolute error of state variable (\( \| x(t) - x_n(t) \|_\infty \)), control variable (\( \| u(t) - u_n(t) \|_\infty \)) and performance index (\( | J - J_n | \)) are listed in Table 2 for different \( n \) of presented algorithm.

Table 2: The maximum absolute error of performance index, state and control variables for different \( n \) in Example 5.1

| \( n \) | \( \| x(t) - x_n(t) \|_\infty \) | \( \| u(t) - u_n(t) \|_\infty \) | \( | J - J_n | \) |
|---|---|---|---|
| 1 | 1.2e^{-1} | 6.3e^{-1} | 5.7e^{-2} |
| 2 | 9.5e^{-3} | 5.5e^{-2} | 1.3e^{-3} |
| 3 | 1.0e^{-3} | 8.0e^{-3} | 2.2e^{-5} |
| 4 | 6.4e^{-5} | 1.5e^{-3} | 1.7e^{-7} |
| 5 | 4.0e^{-6} | 4.8e^{-5} | 8.6e^{-10} |
| 6 | 1.9e^{-7} | 7.6e^{-6} | 2.3e^{-12} |
| 7 | 9.3e^{-9} | 1.6e^{-7} | 7.6e^{-15} |
| 8 | 3.3e^{-10} | 6.8e^{-9} | 1.3e^{-17} |
| 9 | 1.2e^{-11} | 2.5e^{-10} | 2.4e^{-20} |
| 10 | 1.1e^{-11} | 2.3e^{-10} | 2.2e^{-20} |

Example 5.2. \([3, 27]\)

The object is to find the optimal control which minimizes

\[
J = \frac{1}{2} \int_0^2 u^2(t)dt, \quad 0 \leq t \leq 2,
\]

(5.33)

when

\[
u(t) = \dot{x}(t) + \ddot{x}(t),
\]

(5.34)

and

\[
x(0) = 0, \dot{x}(0) = 0, x(2) = 5, \dot{x}(2) = 2,
\]

(5.35)

are satisfied. Where analytical solution is

\[
x(t) = -6.103 + 7.289t + 6.696e^{-t} - 0.593e^t,
\]

and

\[
u(t) = 7.289 - 1.186e^t.
\]
Therefore, the exact value of performance index is \( J = 16.74543860 \). By approximating \( x(t) \) by third order Boubaker series of unknown parameters, we get

\[
x(t) = \sum_{k=0}^{3} a_k B_k(t),
\]

and then the control variable are obtained from the state equation (5.34), as follow:

\[
u(t) = a_1 + 2a_2 + a_3 + (2a_2 + 6a_3) t + 3a_3 t^2,
\]

By substituting (5.37) into (5.33), the following expression for \( \hat{J} \) can be obtained

\[
\hat{J} = [a_1 \ a_2 \ a_3] \begin{bmatrix} 1 & 4 & 11 \\ 4 & \frac{52}{3} & 52 \\ 11 & 52 & \frac{840}{5} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.
\]

From boundary conditions (5.35), another equations representing the initial states are obtained as follow:

\[
\begin{align*}
a_0 + 2a_2 & = 0, \\
a_0 + 2a_1 + 6a_2 + 10a_3 & = 5, \\
a_1 + a_3 & = 0, \\
a_1 + 4a_2 + 13a_3 & = 2.
\end{align*}
\]

The dynamic optimal control problem is approximated by a quadratic programming problem. The new problem is to minimize (5.38) subject to the equality constraint (5.39). The optimal value of the vector \( \mathbf{a}^* \) can be obtained from the standard quadratic programming method. By substituting these optimal parameters into (5.38), the approximate optimal value can be calculated. The optimal cost functional \( J \) obtained by the presented method is shown for different \( n \) in Table 3.

**Table 3: Optimal cost functional \( J \) for different \( n \) in Example 5.2**

<table>
<thead>
<tr>
<th>( n )</th>
<th>( J )</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>16.76304348</td>
<td>1.8e^-2</td>
</tr>
<tr>
<td>5</td>
<td>16.75073345</td>
<td>5.3e^-3</td>
</tr>
<tr>
<td>6</td>
<td>16.75072526</td>
<td>5.2e^-3</td>
</tr>
</tbody>
</table>

For \( n = 5 \) we calculate state and control variables approximately as:

\[
x(t) = 3.05t^2 - 1.19t^3 + 0.218t^4 - 0.0359t^5
\]

and

\[
u(t) = 6.09 - 1.04t - 0.957t^2 + 0.152t^3 - 0.180t^4.
\]

The obtained solution and the analytical solution are plotted in Figure 2.
Figure 2: Solution of Example 5.2. The approximate solution for $n = 5$ is compared with the actual analytical solution.

The accuracy of presented method is determined numerically for the absolute errors $|x(t) - x_n(t)|$ and $|u(t) - u_n(t)|$ as given in Table 4, for $n = 5$.

Table 4: Comparison of exact and approximate solution of $x(t)$ and $u(t)$; for $n = 5$.

<table>
<thead>
<tr>
<th>t</th>
<th>$x(t)$</th>
<th>$x_n(t)$</th>
<th>$u(t)$</th>
<th>$u_n(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.0</td>
<td>0.0</td>
<td>6.103</td>
<td>6.104826804</td>
</tr>
<tr>
<td>0.25</td>
<td>0.1726729717</td>
<td>0.1727230818</td>
<td>5.766145855</td>
<td>5.767699101</td>
</tr>
<tr>
<td>0.50</td>
<td>0.6251375833</td>
<td>0.6253119029</td>
<td>5.333616573</td>
<td>5.334818154</td>
</tr>
<tr>
<td>0.75</td>
<td>1.271335427</td>
<td>1.271672005</td>
<td>4.778237980</td>
<td>4.778987811</td>
</tr>
<tr>
<td>1.00</td>
<td>2.037379614</td>
<td>2.037882844</td>
<td>4.065117752</td>
<td>4.065287338</td>
</tr>
<tr>
<td>1.25</td>
<td>2.856912746</td>
<td>2.857553627</td>
<td>3.149453253</td>
<td>3.148878217</td>
</tr>
<tr>
<td>1.50</td>
<td>3.666937933</td>
<td>3.667652275</td>
<td>1.973716763</td>
<td>1.972185305</td>
</tr>
<tr>
<td>1.75</td>
<td>4.403860939</td>
<td>4.404545299</td>
<td>0.464041226</td>
<td>0.4612816903</td>
</tr>
<tr>
<td>2.00</td>
<td>4.999494789</td>
<td>4.999999996</td>
<td>-1.474420533</td>
<td>-1.478757093</td>
</tr>
</tbody>
</table>

Note that, the previous problem is also solved by expanding $x(t)$ into $7^{th}$ order Boubaker series, and the founded optimal value is $16.75072340$, that is very close to both the exact value $16.74543860$ and the result obtained in [27] (which is $16.74531717$ for three iterations of their algorithms).

**Example 5.3.** (Controlled Linear and Duffing Oscillator [22, 26, 27])

Now we report the approximation of the state and control variables of the controlled linear oscillator problem with the following choice of the numerical values of the parameters in the standard case:

$$\omega = 1, T = 2, x_0 = 0.5, \dot{x}_0 = -0.5,$$

The object is to find the optimal control which minimizes

$$J = \frac{1}{2} \int_{-2}^{0} u(t)^2 dt, \quad -2 \leq t \leq 0,$$  \hspace{1cm} (5.40)

when

$$u(t) = \ddot{x}(t) + x(t),$$  \hspace{1cm} (5.41)

and

$$x(-2) = 0.5, \quad x(0) = 0, \quad \dot{x}(-2) = -0.5, \quad \dot{x}(0) = 0.$$  \hspace{1cm} (5.42)

The approximation of $x(.)$ is considered as follow:

$$x(t) = \sum_{i=0}^{3} a_i B_i(t),$$  \hspace{1cm} (5.43)
and then the control variable are obtained from the state equation (5.41), as follow:

\[ u(t) = 4a_2 + 7a_3t + a_0 + a_1t + a_2t^2 + a_3t^3. \]  

(5.44)

By substituting (5.44) into (5.40), the following expression for \( \hat{J} \) can be obtained

\[
\hat{J} = \begin{bmatrix}
1 & -1 & \frac{32}{5} & -9 \\
-1 & \frac{4}{3} & -6 & \frac{376}{30} \\
\frac{32}{6} & -6 & \frac{448}{15} & -\frac{333}{6} \\
-9 & \frac{376}{30} & -\frac{323}{6} & \frac{12524}{105}
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{bmatrix}.
\]  

(5.45)

From boundary conditions (5.42), another equations representing the initial states are obtained as follow:

\[
\begin{align*}
a_0 - 2a_1 + 6a_2 - 10a_3 &= \frac{1}{2}, \\
a_0 + 2a_2 &= 0, \\
a_1 - 4a_2 + 13a_3 &= -\frac{1}{2}, \\
a_1 + a_3 &= 0.
\end{align*}
\]  

(5.46)

The dynamic optimal control problem is approximated by a quadratic programming problem. The new problem is to minimize (5.45) subject to the equality constraint (5.46). The optimal value of the vector \( a^* \) can be obtained from the standard quadratic programming method. By substituting these optimal parameters into (5.45), the approximate optimal value can be calculated. The optimal cost functional \( J \), obtained by the presented method is shown for different \( n \) in Table 5.

<table>
<thead>
<tr>
<th>n</th>
<th>( J )</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.1849168913</td>
<td>5.8e-3</td>
</tr>
<tr>
<td>5</td>
<td>0.1848735296</td>
<td>1.5e-5</td>
</tr>
<tr>
<td>6</td>
<td>0.1848585740</td>
<td>3.2e-8</td>
</tr>
</tbody>
</table>

Table 5: Optimal cost functional \( J \) for different \( n \) in Example 5.3

The exact solution for the performance index is \( J = 0.1848585422 \). The optimal cost functional, \( J \), obtained by the presented method for \( n = 6 \) is a very accurate approximation of the exact solution. This leads to state and control variables approximately as:

\[
x(t) = 0.0125t^2 - 0.0895t^3 + 0.00184t^4 + 0.0127t^5 + 0.00174t^6
\]

and

\[
u(t) = 0.0250 - 0.537t + 0.0346t^2 + 0.165t^3 + 0.0540t^4 + 0.0127t^5 + 0.00174t^6
\]

The obtained solution and the analytical solution are plotted in Figure 3.
Figure 3: Solution of Example 5.3. The approximate solution for \( n = 6 \) is compared with the actual analytical solution.

Also, the previous problem is solved by expanding \( x(t) \) into 10th order Boubaker series, and the optimal value is found to be 0.1848585424 which is very close to the exact value 0.1848585422. The result obtained in [26] and [27] are 0.184858576 and 0.18497926 for three iterations of their algorithms. The maximum absolute error of state variable (\( \|x(t) - x_n(t)\|_\infty \)), control variable (\( \|u(t) - u_n(t)\|_\infty \)) and performance index (\( |J - J_n| \)) are listed in Table 6 for different \( n \) of presented algorithm.

| \( n \) | \( \|x(t) - x_n(t)\|_\infty \) | \( \|u(t) - u_n(t)\|_\infty \) | \( |J - J_n| \) |
|---|---|---|---|
| 2 | 3.3e^{-2} | 1.1e^{-1} | 1.1e^{-2} |
| 3 | 3.2e^{-2} | 1.0e^{-1} | 1.0e^{-2} |
| 4 | 6.9e^{-4} | 7.7e^{-3} | 5.8e^{-5} |
| 5 | 3.0e^{-4} | 5.2e^{-3} | 1.5e^{-5} |
| 6 | 6.9e^{-6} | 1.9e^{-4} | 3.2e^{-8} |
| 7 | 1.1e^{-6} | 7.0e^{-5} | 2.3e^{-9} |
| 8 | 3.3e^{-8} | 2.2e^{-6} | 2.2e^{-10} |
| 9 | 2.6e^{-9} | 3.4e^{-7} | 2.1e^{-10} |
| 10 | 9.5e^{-10} | 8.5e^{-9} | 2.0e^{-10} |

The controlled Duffing Oscillator

Now, we investigate the optimal controlled Duffing oscillator. As mentioned before, the exact solution in this case is not known. Table 7 lists the optimal values of the cost functional \( J \) for various values of \( \epsilon \) for different \( n \) for controlled Duffing Oscillator.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \epsilon = 0.15 )</th>
<th>( \epsilon = 0.5 )</th>
<th>( \epsilon = 0.75 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.1992829956</td>
<td>0.2075425443</td>
<td>0.2136225103</td>
</tr>
<tr>
<td>4</td>
<td>0.1875292166</td>
<td>0.1937080476</td>
<td>0.1981923010</td>
</tr>
<tr>
<td>5</td>
<td>0.1874970425</td>
<td>0.1936303211</td>
<td>0.1980668596</td>
</tr>
</tbody>
</table>
6 Conclusion

This paper presents a numerical technique for solving nonlinear optimal control problems and the controlled Duffing oscillator as a special class of optimal control problems. The solution is based on state parametrization. It produces an accurate approximation of the exact solution, by using a small number of unknown coefficients. We emphasize that this technique is effective for all classes of optimal control problems. In fact, the direct method proposed here has potential to calculating continuous control and state variables as functions of time. Also, the numerical value of the performance index is obtained readily. This method provides a simple way to adjust and obtain an optimal control which can easily be applied to complex problems as well. The convergence of the algorithms is proved. One of the advantages of this method is its fast convergence. Some illustrative examples are solved by this method, the results show that the presented method is a powerful method, which is an important factor to choose the method in engineering applications.

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