Three-step iterative methods with eighth-order convergence for solving nonlinear equations

M. Matinfar 1; M. Aminzadeh 2

(1) Department of Mathematics, University of Mazandaran, P.O.Box 47415-95447, Babolsar, Iran
(2) Science of Mathematics Faculty, Department of Mathematics, University of Mazandaran

Copyright 2013 © M. Matinfar and M. Aminzadeh. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

A family of eighth-order iterative methods for solution of nonlinear equations is presented. We propose an optimal three-step method with eight-order convergence for finding the simple roots of nonlinear equations by Hermite interpolation method. Per iteration of this method requires two evaluations of the function and two evaluations of its first derivative, which implies that the efficiency index of the developed methods is \( \sqrt[3]{682} \). Some numerical examples illustrate that the algorithms are more efficient and performs better than the other methods.

Keywords: Nonlinear equation; Iterative method; Three-step; Convergence order; Efficiency index.

1 Introduction

In this paper, we develop an iterative method to find a simple root \( \alpha \) of the nonlinear equation \( f(\alpha) = 0 \), where \( f : D \subset \mathbb{R} \rightarrow \mathbb{R} \) is a scalar function on an open interval \( D \). It is well known that Newton’s method is one of the best iterative methods for solving a single nonlinear equation by using

\[
 x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},
\]

which converges quadratically in some neighborhood of \( \alpha \). Ostrowski’s method[20], given by

\[
 \begin{align*}
 y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
 x_{n+1} &= y_n - \frac{f(y_n)}{f'(y_n)} \cdot \frac{f(y_n)}{f'(x_n)},
\end{align*}
\]

is an improvement of Newton’s method. The order increases by at least two at the expense of additional function evaluation at another point iterated by the Newton’s method. To improve the local order of convergence and efficiency index, many more modified methods have been proposed in open literatures, see [2 – 21] and references therein.

Chun and Ham developed a family of variants of Ostrowski’s method with sixth-order methods by weight function
methods in [7] (see (12) – (17) therein), which is written as:

\[
\begin{align*}
\gamma_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
\delta_n &= y_n - \frac{f(y_n)}{f'(y_n)}, \\
x_{n+1} &= \frac{f(x_n) f'(y_n) - f(y_n) f'(x_n)}{f'(y_n) f'(x_n)},
\end{align*}
\]

where \( \mu_n = \frac{f(x_n)}{f'(x_n)} \) and \( H(t) \) represents a real-valued function with \( H(0) = 1, H'(0) = 2 \) and \( |H''(0)| < \infty. \)

Kou et al. presented a family of variants of Ostrowski’s method [18] (see (23) therein) with seventh-order convergence, which is given by:

\[
\begin{align*}
\gamma_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
\delta_n &= y_n - \frac{f(y_n)}{f'(y_n)} - 2f(x_n) f'(y_n) f'(x_n) f''(y_n) + f(y_n), \\
x_{n+1} &= z_n - \frac{f(z_n) f'(y_n)^2 - f(y_n) f'(x_n)^2}{f'(y_n) f'(x_n)},
\end{align*}
\]

where \( \alpha \) is constant.

Bi et al. presented a family of eighth-order convergence methods [1] (see (5) therein), which is given by:

\[
\begin{align*}
\gamma_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
\delta_n &= y_n - \frac{2f(x_n) - 2f(y_n)}{2f(x_n) - 2f(y_n) f'(x_n)} f'(y_n), \\
x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n) f'(y_n)} + f(z_n, x_n, y_n),
\end{align*}
\]

where \( \mu_n = \frac{f(x_n)}{f'(x_n)} \) and \( H(t) \) represents a real-valued function with \( H(0) = 1, H'(0) = 2 \) and \( |H''(0)| < \infty. \)

Recently Bi et al. presented a new family of eighth-order iterative methods [2] (see (24) therein), which is given by:

\[
\begin{align*}
\gamma_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
\delta_n &= y_n - \frac{h(x_n) f'(y_n)}{f'(x_n)} f''(y_n), \\
x_{n+1} &= z_n - \frac{f(z_n) + (\gamma + 2)f(x_n)}{f'(x_n) f'(y_n) f''(y_n) + f(z_n, x_n, y_n) + f(z_n, x_n, y_n)(z_n - x_n)},
\end{align*}
\]

where \( \gamma \in \mathbb{R} \) is constant, \( \mu_n = \frac{f(x_n)}{f'(x_n)} \) and \( H(t) \) represents a real-valued function with \( H(0) = 1, H'(0) = 2, H''(0) = 10 \) and \( |H'''(0)| < \infty. \) Recently, there are several eighth-order methods proposed in [7 – 9]. Now after furnishing the outlines of the present work and a short study on the available high order developments of the classical Newtons method, we will provide our contribution in the next section. In section 2, we give a general class of efficient three-step eighth-order methods including two evaluations of the function and two of its first derivative per cycle. In section 3, we give some concrete iterative forms. In section 4, the numerical comparisons are made to manifest the accuracy of the new methods from our class. Finally, the conclusion of the paper will be drawn in section 5.

# 2 Development of method and convergence analysis

To develop the new method, let us consider the iteration scheme in the form

\[
\begin{align*}
y_n &= x_n - \frac{m}{m+1} \frac{f(x_n)}{f'(x_n)}, \\
z_n &= y_n - \left[ n + \frac{f'(x_n)^2 f'(y_n)^2 - f(x_n) f'(y_n)^2}{k f'(y_n)^2 + f'(x_n)^2} \right] \frac{f(x_n)}{f'(y_n)},
\end{align*}
\]

where \( m, n, l, k \) are the disposable parameters. For the iterative method defined by (7), we have the following convergence result.
Theorem 2.1. Let $x^* \in D$ be a simple zero of a sufficiently differentiable function $f : D \rightarrow R$ for an open interval $D$. If $x_0$ is sufficiently close to $x^*$, and $m, n, l, k$ satisfy the condition: \{ $m = 2, n = 1 \frac{1}{25}, k = \frac{294}{125}, l = \frac{126}{125}$ \} then the method defined by (2.7) is of fourth-order.

**Proof 2.1.** Let $x^*$ be a simple zero of $f$. Consider the iteration function $F$ defined by

\[
F(x) = y(x) - \left[ n + \frac{1}{k} \frac{f'(x)^2}{f(x)} \right] \frac{f(x)}{f'(y(x))},
\]

In view of an elementary, tedious evaluation of derivatives of $F$, we employ the symbolic computation of the Maple package to compute the Taylor expansion of $F(x_n)$ around $x = x^*$. We find after simplifying that

\[
x_{n+1} = F(x_n) = \alpha - \frac{g_1}{(m+1)(k+l)n} \epsilon_n - \frac{g_3}{(k+l)^2(n+1)n} \epsilon_n^2,
\]

where

\[
g_1 = (m+1)(k+l)n - l - k + m + 1, \quad g_2 = -l^2m - 2lkm - k^2m + nl^2m - nl^2 + 2nlkm - 2nlk
\]

\[+ nk^3m - nk^2l + lm - l + 5km - k
\]

\[g_3 = -(k+l)(-2k^2m - 2k^2m^2 + 2nk^2m^2 + 2nk^2m^2 - 2nk^2
\]

\[+ 14km - 4lkm - 4lkpm + 4nlkm - 4nlk + 7km^2
\]

\[-2k + 2nl^2m - 2nl^2m + 2lm + 2lm + nl^2m - 2l - 2l^2m - 2l^2m
\]

\[g_4 = ((-2 + 2n)k^3 + ((-2 + (6n - 6))k^2 + ((6n - 6)l^2 + 16l)k
\]

\[+ (-2 + 2nl)^3 + 2l^2m^2 + (4((-1/2 + n)^2 + (5 + (-1 + 2n))l
\]

\[+ (-1/2 + 2n))^2 + l)))(k+l)m - 2(k+l)^2(nl + nk + 1)
\]

Solving system of the equations \{ $g_1 = 0, g_2 = 0, g_3 = 0, g_4 = 0$ \} we find that \{ $m = 2, n = 1 \frac{1}{25}, k = \frac{294}{125}, l = \frac{126}{125}$ \} thereby we obtain the new fourth-order iterative method

\[
\begin{align*}
y_n &= x_n - \frac{2}{3} f'(x_n)^2 \frac{f(x_n)}{f'(y_n)} \\
z_n &= y_n - \left[ \frac{1}{125} + \frac{1}{30} \frac{f'(x_n)^2}{f'(y_n)^2} \right] \frac{f(x_n)}{f'(y_n)}
\end{align*}
\]

and satisfies the following error equation:

\[
\epsilon_{n+1} = \frac{1}{9} c_4 - c_2 c_3 + \frac{721}{405} \epsilon_n^3 + O(\epsilon_n^5).
\]

Now we construct a three-step iterative method

\[
\begin{align*}
y_n &= x_n - \frac{2}{3} f'(x_n)^2 \frac{f(x_n)}{f'(y_n)} \\
z_n &= y_n - \left[ \frac{1}{125} + \frac{1}{30} \frac{f'(x_n)^2}{f'(y_n)^2} \right] \frac{f(x_n)}{f'(y_n)} \\
x_{n+1} &= z_n - H(\mu_n) \frac{f(z_n)}{f'(z_n)}, \quad \mu_n = \frac{1}{z_n - f'(z_n)}
\end{align*}
\]

International Scientific Publications and Consulting Services
where $H(\mu_n)$ represents a real-valued function.

We construct an optimal efficiency index with an eight-order convergence for finding the simple roots of nonlinear equations by Hermite interpolation method. We can express $f'(z_n)$ as follows: by using the Hermite interpolation on three points $(x_n, f(x_n)), \ (y_n, f'(y_n))$ and $(z_n, f(z_n), f'(z_n))$ we can approximate $f'(z_n)$ if we solve the equations (2.14).

\[
\begin{align*}
0 + a_1 + 2a_2z_n &= a_3 f'(z_n) \\
0 + a_1 z_n + a_2 z_n^2 &= a_3 f(z_n) \\
0 + a_1 + 2a_2 y_n &= a_3 f(y_n) \\
a_0 + a_1 x_n + a_2 x_n^2 &= a_3 f(x_n)
\end{align*}
\] (2.14)

for the coefficients $a_i, 0 \leq i \leq 3$ and $a_3 = 1$.

The system (2.14) has a unique solution if and only if the Determinant($M_z$) = 0,

\[
M_z = \begin{bmatrix}
0 & 1 & 2z_n & -f'(z_n) \\
1 & z_n & z_n^2 & -f(z_n) \\
0 & 1 & 2y_n & -f(y_n) \\
1 & y_n & y_n^2 & -f(x_n)
\end{bmatrix}
\]

by expanding Determinant($M_z$) about fourth column we obtain

\[
(A)f'(z_n) - (B)f(z_n) + (C)f(y_n) - (D)f(x_n) = 0,
\] (2.15)

As, we have

\[
f'(z_n) = \frac{(B)}{(A)} f(z_n) - \frac{(C)}{(A)} f(y_n) + \frac{(D)}{(A)} f(x_n),
\] (2.16)

where

\[
\begin{align*}
A &= (x_n - z_n)(x_n - 2y_n + z_n), \\
B &= 2(y_n - z_n), \\
C &= -(x_n - z_n)^2, \\
D &= -2(y_n - z_n),
\end{align*}
\] (2.17)

Substituting "A, B, C, D" of (2.17) in (2.16) and simplifying "\[ \frac{(B)}{(A)}, \frac{(C)}{(A)}, \frac{(D)}{(A)}\]" we have $f'(z_n) \approx \Psi_f(x_n, y_n, z_n)$:

\[
\begin{align*}
\Psi_f(x_n, y_n, z_n) &= \frac{2(y_n - z_n)f(z_n)}{(x_n - z_n)(x_n - 2y_n + z_n)} + \frac{(y_n - z_n)f'(y_n)}{(x_n - z_n)(y_n - 2y_n + z_n)} \\
&\quad + \frac{2(y_n - z_n)f(x_n)}{(x_n - z_n)(x_n - 2y_n + z_n)}.
\end{align*}
\] (2.18)

By simplifying, we have

\[
\Psi_f(x_n, y_n, z_n) = \frac{(z_n - x_n)}{x_n - 2y_n + z_n} \left[ \frac{(z_n - x_n)}{z_n - x_n} f(z_n, x_n) - f'(y_n) \right],
\] (2.19)

where $f[z_n, x_n] = \frac{f(z_n) - f(x_n)}{z_n - x_n}$. 


By replacing \( f'(z_n) \) with approximation in (2.13), we construct a three-step iterative method

\[
\begin{align*}
    y_n &= x_n - \frac{2}{3} \frac{f(x_n)}{f'(x_n)}, \\
    z_n &= y_n - \frac{1}{3} + \frac{2}{93} \frac{f'(x_n)^2}{f(x_n)} \\
    x_{n+1} &= z_n - H(\mu_n) \frac{f(z_n)}{f'(x_n)}, \quad \mu_n = \frac{1}{2 n - \Psi_f(x_n, y_n, z_n)},
\end{align*}
\]

(2.20)

where \( H(t) \) represents a real-valued function. We prove the following convergence theorem for the method (2.20).

**Theorem 2.2.** Assume that functions \( H, f \) are sufficiently differentiable and \( f \) has a simple zero \( x^* \in D \). If the initial point \( x_0 \) is sufficiently close to \( x^* \), then the method defined by (2.20) converges to \( x^* \) with eighth-order under the conditions \( "H(-1) = 1, H'(-1) = 1 \) and \( |H''(-1)| < \infty" \).

**Proof 2.2.** Let \( e_n = x_n - x^* \) and \( c_k = \frac{f^{(k)}(x^*)}{k! f'(x^*)} \) for \( k = 2, 3, \ldots \) By expanding \( f(x_n) \) and \( f'(x_n) \) about \( x^* \), we have

\[
f(x_n) = f'(x^*) (e_n + \sum_{i=2}^{8} c_i e_i + O(e_n^9)),
\]

(2.21)

and

\[
f'(x_n) = f'(x^*) (1 + \sum_{i=2}^{8} i c_i e_i^{i-1} + O(e_n^8)),
\]

(2.22)

by expanding \( y_n \) about \( x_n \), we obtain

\[
y_n = x^* + \frac{1}{3} e_n + \frac{2}{3} c_2 e_n^2 + \left( \frac{4}{3} c_3 - \frac{4}{3} c_2^2 \right) e_n^3 + \left( 2 c_4 - \frac{14}{3} c_2 c_3 + \frac{8}{3} c_3^2 \right) e_n^4 + \cdots + O(e_n^9),
\]

(2.23)

furthermore, by expanding \( f'(y_n) \) about \( x_n \), we have

\[
f'(y_n) = f'(x^*) (1 + \frac{2}{3} c_2 e_n + \frac{4}{3} c_2 c_3 + \frac{1}{3} c_3^2 e_n^2 + \left( 4 c_2 c_3 - \frac{8}{3} c_3^2 + \frac{4}{27} c_4 \right) e_n^3 + \cdots + O(e_n^9)).
\]

(2.24)

Moreover, we find

\[
\begin{align*}
    z_n &= x^* - \left( \frac{1}{2} c_4 - c_2 c_3 + \frac{721}{1058} c_3^2 \right) e_n^4 + \cdots + \left( \frac{-837619}{2025} c_4 c_3 c_2^2 \right) e_n^8 + O(e_n^9) \\
    &\quad + \frac{634491}{6075} c_2 c_3^2 + \frac{49279}{1215} c_4 c_3^2 + \frac{2546722}{10935} c_4 c_2^2 - \frac{349013}{2025} c_2 c_3^2 \\
    &\quad + \frac{1290316}{6075} c_5 c_2^2 - \frac{177022042}{455625} c_3 c_2^5 + \frac{152662544}{2278125} c_7 e_n^8 + O(e_n^9).
\end{align*}
\]

(2.25)

We similarly find the expansion of \( f(z_n) \).

\[
\begin{align*}
    f(z_n) &= f'(x^*) \left( \frac{1}{2} c_4 - c_2 c_3 + \frac{721}{1058} c_3^2 \right) e_n^4 + \cdots - \frac{838069}{2025} c_4 c_3 c_2^2 \\
    &\quad + \frac{634276}{10935} c_2 c_3^2 + \frac{49279}{1215} c_4 c_3^2 + \frac{4646048}{10935} c_4 c_2^2 - \frac{349013}{2025} c_2 c_3^2 \\
    &\quad + \frac{339091}{6075} c_5 c_2^2 - \frac{178444292}{455625} c_3 c_2^5 + \frac{1438943021}{20503125} c_7 e_n^8 + O(e_n^9)
\end{align*}
\]

(2.26)
By expanding $\Psi_f(x_n, y_n, z_n) \approx f'(z_n)$ about $x_n$, we have

$$
\begin{align*}
\Psi_f(x_n, y_n, z_n) &= f'(x^*) (1 - c_3 e_n^2 + \left(-\frac{14}{7} c_4 - 4c_2 c_3\right) e_n^2 + \cdots) \\
&= -76348 \cdot 10^{-15} c_4 c_3^2 + 896 \cdot 10^{-9} c_2 c_3^2 - 1660 \cdot 10^{-6} c_1 c_3^2 - 1146496 \cdot 10^{-5} c_4 c_2^2 \\
&\quad - 56432 \cdot 10^{-3} c_2^3 + 967124 \cdot 10^{-2} c_3^3 + 1091428 \cdot \frac{1}{100375} c_3 c_2^2 - 127309496 \cdot \frac{1}{1056875} c_2^3 e_n^2 + O(e_n^8). 
\end{align*}
$$

(2.27)

By substituting (2.25), (2.26) and (2.27) into the third formula of (2.20), using Taylor’s expansion, and simplifying, we have Thus, we have

$$
x_{n+1} = \left(-\frac{1}{305} (45c_4 - 405c_2c_3 + 721c_3^2)\right) (1 + H(-1)) e_n^4 + \left(\frac{2}{2025} (2250c_2c_4 + 2025c_2^2 - 12840c_3 c_2^2 + 8402c_4^2)\right) (-1 + H(-1)) e_n^5 + \left(-\frac{702758}{30375} H(-1)c_2^2\right) + 65 H(-1)c_4c_3 - \frac{23060}{105} H(-1)c_2c_3^2 + \frac{13033}{225} H(-1)c_1c_3^2 \\
+ 6 H(-1)c_2c_4 - 2 - 211 \cdot 10^{-3} H(-1)c_2^2c_3 + 23060 \frac{1}{105} c_4c_3^2 + 3694 c_2c_3^2 \\
- \frac{129002}{2025} c_2 c_3^3 + \frac{702758}{30375} c_2^3 e_n^6 + \left(\frac{27124}{1442} c_4 c_2 c_3 - \frac{324}{519841} c_2^4 - \frac{1469896}{104672} c_4 c_2^3 - \frac{104672}{675} c_3 c_2^2\right). 
$$

(2.28)

By substituting $H(-1) = 1$ in (2.28), and simplifying, we have

$$
x_{n+1} = \left(-\frac{1}{305} (45c_4 - 405c_2c_3 + 721c_3^2)\right) c_3 (H'(-1) - 1) e_n^6 \\
\quad - \left(\frac{2}{18225} (-79110c_2 c_3^2 + 10728c_2^2 c_4 + 30375c_4c_2c_3 - 1575c_3^2)\right) (H'(-1) - 1) c_4^2 + \frac{81}{10} (H'(-1)c_2 c_3^2) e_n^6 + O(e_n^8). 
$$

(2.29)

By substituting $H'(1) = 1$ in (2.29), and simplifying, we have

$$
x_{n+1} - x^* = O(e_n^8). 
$$

(2.30)

and satisfies the following error equation :
which shows that the order of convergence of our new proposed method defined in (2.20) is eight. This completes the proof.

3 The concrete iterative methods

In what follows, we give some concrete iterative forms of (2.20).

Method 3.1. For the function $G$ defined by

$$G_i(t) = \frac{t^{2i-1} + 2t}{2i-1},$$

(3.32)

where $i \in \mathbb{Z}$, it can easily be seen that the function $G_i(t)$ of (3.32) satisfies conditions of Theorem 2.2. Hence we get a new one-parameter eighth-order family of methods

$$
\begin{align*}
    y_n &= x_n = \frac{2f(x_n)}{f'(x_n)}, \\
    z_n &= y_n = \frac{1}{28} + \frac{f'(x_n)^2}{28f'(x_n)^2 + 125f''(x_n)^2} f(x_n) \\
    x_{n+1} &= z_n = \sqrt{\frac{1}{z_n} \Psi_f(x_n, y_n, z_n)} + 1 \\
\end{align*}
$$

(3.33)

Method 3.2. For the function $H$ defined by

$$H(t) = \sin(t + 1) + 1,$$

(3.34)

it can easily be seen that the function $H(t)$ of (3.34) satisfies conditions of Theorem 2.2. We get another new eighth-order methods

$$
\begin{align*}
    y_n &= x_n = \frac{2f(x_n)}{f'(x_n)}, \\
    z_n &= y_n = \frac{1}{28} + \frac{f'(x_n)^2}{28f'(x_n)^2 + 125f''(x_n)^2} f(x_n) \\
    x_{n+1} &= z_n = \sin\left(\frac{1}{z_n} \Psi_f(x_n, y_n, z_n) + 1\right) + 1 \\
\end{align*}
$$

(3.35)

Method 3.3. For the function $K$ defined by

$$K(t) = e^{t+1},$$

(3.36)

it can easily be seen that the function $K(t)$ of (3.36) satisfies conditions of Theorem 2.2. We get another new eighth-order methods

$$
\begin{align*}
    y_n &= x_n = \frac{2f(x_n)}{f'(x_n)}, \\
    z_n &= y_n = \frac{1}{28} + \frac{f'(x_n)^2}{28f'(x_n)^2 + 125f''(x_n)^2} f(x_n) \\
    x_{n+1} &= z_n = \left(\frac{1}{z_n} \Psi_f(x_n, y_n, z_n) + 1\right) f(x_n) \\
\end{align*}
$$

(3.37)

Remark 3.1. The order of convergence of the iterative methods (3.33), (3.35) and (3.37) is 8. Per iteration the presented methods requires two evaluations of the function, namely, $f(x_n)$ and $f(z_n)$ and two evaluations of first derivatives $f'(x_n)$ and $f'(y_n)$. We take into account the definition of efficiency index [11] as $p^{1/w}$, where $p$ is the order of the method and $w$ is the number of function evaluations per iteration required by the method. If we suppose that all the evaluations have the same cost, we have that the efficiency index of the new methods is $\sqrt{8} \approx 1.682$, which...
we used the fixed stopping criterion when the following functions: 

$$LWM$$

$$\gamma \in \mathbb{R}.$$ Liu and Wang in [19] presented the following family of optimal order eight methods. Sharma in [21] to produce optimal eighth-order method in the following form

$$\text{Bi}$$'s method, Liu's method and Sharma's method with

$$8$$-th order convergence.

$$\text{COC}$$

$$\text{Bi}$$ et al, developed a scheme of optimal order of convergence eight[2], estimating the first derivative of the function in the second and third steps and constructing a weight function as well in the following form

$$\text{BM}$$

$$\text{LWM}$$

$$\text{ShM}$$

$$\text{Bi}$$ et al, developed a scheme of optimal order of convergence eight[2], estimating the first derivative of the function in the second and third steps and constructing a weight function as well in the following form

$$\text{BM}$$

$$\text{LWM}$$

$$\text{ShM}$$

We present some numerical results to illustrate the efficiency of the three-step iterative method proposed in this paper. We compare (3.33), (3.35) and (3.37) with BM, LWM and ShM. All computations were done using Matlab. We use the following stopping criteria for computer programs: 

$$\left| x_{n+1} - x_n \right| < \varepsilon, \quad \left| f(x_{n+1}) \right| < \varepsilon$$

and so, when the stopping criterion is satisfied, 

$$x_{n+1}$$

is taken as a computed value of the exact root. For numerical illustrations in this section we used the fixed stopping criterion 

$$\varepsilon = 10^{-15},$$

where 

$$\varepsilon$$

represents tolerance. We present some numerical test results with the following functions:

$$f_1(x) = x^3 + 4x^2 - 15,$$

$$f_2(x) = x^3 - x^2 + 4x - 15,$$

$$f_3(x) = x^3 - x^2 + 4x - 15,$$

$$f_4(x) = x^3 - x^2 + 4x - 15,$$

$$f_5(x) = x^3 - x^2 + 4x - 15,$$

$$f_6(x) = x^3 - x^2 + 4x - 15,$$

$$f_7(x) = x^3 - x^2 + 4x - 15,$$

$$f_8(x) = x^3 - x^2 + 4x - 15,$$

$$f_9(x) = x^3 - x^2 + 4x - 15,$$

$$f_{10}(x) = x^3 - x^2 + 4x - 15,$$

$$f_{11}(x) = x^3 - x^2 + 4x - 15,$$

$$f_{12}(x) = x^3 - x^2 + 4x - 15,$$

$$f_{13}(x) = x^3 - x^2 + 4x - 15,$$

$$f_{14}(x) = x^3 - x^2 + 4x - 15.$$
In this work we presented an approach which can be used to constructing of eight-order convergence iterative methods that do not require the computation of second or higher derivatives. Numerical examples also show that the numerical results of our new methods, in equal iterations, improve the results of other existing three-step methods with eight-order convergence. Finally, it is hoped that this study makes a contribution to solve nonlinear equations.

## References

http://dx.doi.org/10.1016/j.cam.2008.07.004

http://dx.doi.org/10.1016/j.amc.2009.03.077
http://dx.doi.org/10.1016/j.amc.2006.11.127

http://dx.doi.org/10.1016/j.amc.2007.04.105

http://dx.doi.org/10.1016/j.amc.2007.02.023

http://dx.doi.org/10.1016/j.camwa.2008.05.005

http://dx.doi.org/10.1016/j.amc.2007.03.074

http://dx.doi.org/10.1016/j.amc.2007.05.003

http://dx.doi.org/10.1016/j.mcm.2010.05.033

http://dx.doi.org/10.1016/j.cam.2006.11.022


http://dx.doi.org/10.1016/j.amc.2007.04.005

http://dx.doi.org/10.1016/j.cam.2007.11.018

http://dx.doi.org/10.1016/j.amc.2006.11.117

http://dx.doi.org/10.1016/j.amc.2006.09.097

http://dx.doi.org/10.1016/j.amc.2006.10.018

http://dx.doi.org/10.1016/j.amc.2006.12.062
http://dx.doi.org/10.1016/j.cam.2006.10.073

http://dx.doi.org/10.1016/j.amc.2009.10.040


http://dx.doi.org/10.1007/s11075-009-9345-5