Solution of Two-dimensional Fredholm Integral Equation via RBF-triangular Method

Amir Fallahzadeh *

Department of mathematics, Islamic Azad University Central Tehran Branch, P.O. Box 13185.768, Tehran, Iran.

Abstract
In this paper, a new method is introduced to solve a two-dimensional Fredholm integral equation. The method is based on the approximation by Gaussian radial basis functions and triangular nodes and weights. Also, a new quadrature is introduced to approximate the two-dimensional integrals which is called the triangular method. The results of the example illustrate the accuracy of the proposed method increases.

Keywords: Fredholm integral equation; Two-dimensional integral equation; Radial basis functions (RBFs); Triangular nodes.

1 Introduction

In recent years, some numerical methods have been proposed to estimate the solution of one-dimensional and two-dimensional integral equations such as [1, 2, 7, 8]. RBFs played an important role in approximation theory to introduce a new basis in numerical solution of integral equations [3, 5, 9, 10]. In this work, the Gaussian radial basis functions (RBFs) is applied to solve the two-dimensional Fredholm integral equation of the second kind as follows,

\[ u(x, y) = f(x, y) + \int_a^b \int_c^d k(x, y, s, t)u(s, t)dsdt, \quad (s, t) \in D, \quad (1.1) \]

where \( f(x, y) \) and \( k(x, y, s, t) \) are given continuous functions defined respectively on \( D = [a, b] \times [c, d] \), \( E = D \times D \) and \( u \) is unknown on \( D \), and the corresponding double integrals are approximated by the new method which is called the triangular method.

*Corresponding author. Email address: amir_falah6@yahoo.com, Tel: +982188385773
In section 2, the RBFs and properties of them is introduced. In section 3, we offer the triangular quadrature method. In section 4, the RBF-triangular method is proposed to obtain the solution of the two-dimensional Fredholm integral equation (1.1) and in section 5, a numerical example is solved and compared with the results of the method in [7].

2 Preliminaries

**Definition 2.1.** The function $\phi$ on $X$ is said to be positive definite on $X$ if for any finite set of points $x_1, x_2, \ldots, x_n$ in $X$ the $n \times n$ matrix $A_{ij} = \phi(x_i - x_j)$ is nonnegative definite, i.e.,

$$u^* Au = \sum_{i=1}^{n} \sum_{j=1}^{n} u_i u_j A_{ij} \geq 0$$

for all $u \in \mathbb{C}^n$. If $u^* Au > 0$ whenever the points $x_i$ are distinct and $u \neq 0$, then we say that $\phi$ is strictly positive definite. [4, 6]

**Theorem 2.1.** (Bochner’s theorem) [4]. Let $f$ be a nonnegative Borel function on $\mathbb{R}$, if $0 < \int_\mathbb{R} f < \infty$, then $\hat{f}$ is strictly positive definite, where $\hat{f}$ is the Fourier transform of function $f$, i.e.

$$\hat{f} = \int_{-\infty}^{+\infty} f(y)e^{ixy}dy.$$

A real-value function $F$ on an inner-product space is said to be radial if $F(x) = F(y)$ whenever $\|x\| = \|y\|$. Now by using above theorem, we can find many strictly positive definite functions or (RBFs). For example if $f(x) = e^{-|x|^2}$ then $\hat{f}(x) = \frac{1}{\sqrt{\pi^2}}$.

Radial basis functions were introduced in [4], and we use these basis functions to approximate the function $F(x, y)$. Let $\phi(r)$ be a RBF, we know that it interpolates data $((x_k, y_k), z_k)$ for $k = 0, 1, \ldots, n$ uniquely. Therefore:

$$F(x, y) \approx \sum_{k=0}^{n} c_k \phi(\|(x, y) - (x_k, y_k)\|)$$

where: $\|(x, y) - (x_k, y_k)\| = (x - x_k)^2 + (y - y_k)^2)^{\frac{1}{2}}$, and coefficients $c_k$ are determined in such way that: $F(x_k, y_k) = z_k$, $k = 0, 1, \ldots, n$.

3 Introducing triangular quadrature

In this case, a new quadrature method is used to approximate a two-dimensional integral. For this purpose, we consider the two-dimensional basis functions $1, x, y, xy, x^2, y^2, \ldots$, and calculate the weights such that the method is exact for these basis functions. We know there is no Haar space in $\mathbb{R}^2$, therefore we must consider the node points conditionally. For this purpose, we consider the nodes in shape of triangle, (see [4], section 10) and the following quadrature is introduced:

$$\int_{a}^{b} \int_{c}^{d} f(x, y)dydx = \int_{a}^{b} \int_{c}^{d} f(x, y)dydx + \int_{a}^{b} \int_{d}^{l} f(x, y)dydx \simeq$$
for $i, j$ in Eq. (4.6) and hence this equation can be written as follows:

$$\sum_{i=0}^{n} \sum_{j=0}^{n-i} w_{ij} f(x_i, y_j) + \sum_{i=0}^{n} \sum_{j=0}^{n-i} w_{ij} f(x_{n-j}, y_{n-i}),$$

(3.2)

where $l = \frac{d-c}{a-b}(x-a) + d$ and $x_i = a + ih, y_i = b + ik$ ($i = 0, 1, \ldots, n$) and $h = \frac{(b-a)}{n}, k = \frac{(d-c)}{n}$, we call these points triangle nodes. The weights $w_i$ are calculated numerically. For example if $n = 2$ then the weights are:

$$w_{00} = \frac{1}{6} \left\{ \frac{(c-d)}{(a-b)} (a-d-c+d)^2, \right.$$  
$$w_{01} = \frac{1}{6} \left\{ \frac{(c-d)^2}{(a-b)} (3a-3b-2c+2d), \right.$$  
$$w_{02} = \frac{1}{6} \left\{ \frac{(c-d)^2}{(a-b)(a-d-c+d)}, \right.$$  
$$w_{10} = \frac{1}{6} \left\{ (c-d)(2a-2b-c+d), \right.$$  
$$w_{11} = \frac{1}{6} \left\{ (c-d)^2, \right.$$  
$$w_{20} = 0,$$

respected to the basis functions: $1, x, x^2, y, xy, y^2$.

4 Main idea

In this section, the RBFs are used to approximate the solution of the following integral equation by discretizing the two-dimensional integral via triangular nodes.

$$u(x, y) = f(x, y) + \int_{a}^{b} \int_{c}^{d} k(x, y, s, t) u(s, t) ds dt, \quad (s, t) \in D.$$  

(4.3)

Let $\phi(r)$ be a strictly definite function or RBF, then we approximate $u(x, y)$ by the function $\phi(r)$, i.e.

$$u(x, y) \simeq \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij} \phi_{ij}(x, y),$$

(4.4)

where, $\phi_{ij}(x, y) = \phi(||(x, y) - (x_i, y_j)||)$.

Also $(x_i, y_j)$ are the node points. Now by substituting Eq. (4.4) in Eq. (4.3) we have:

$$\sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij} \phi_{ij}(x, y) \simeq f(x, y) + \int_{a}^{b} \int_{c}^{d} k(x, y, s, t) \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij} \phi_{ij}(s, t) ds dt.$$  

(4.5)

For obtaining $c_{ij}, i, j = 0, 1, \ldots, n$ in the Eq. (4.5), we consider the points $(x, y) = (x_i, y_j)$ for $i, j = 0, 1, \ldots, n$, where that $(x_i, y_j)$ are triangular nodes, we have:

$$\sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij} \phi_{ij}(x_i, y_j) = f(x_i, y_j) + \int_{a}^{b} \int_{c}^{d} k(x_i, y_j, s, t) \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij} \phi_{ij}(s, t) ds dt.$$  

(4.6)

By applying the triangular method mentioned in section 3, we can approximate the integral in Eq. (4.6) and hence this equation can be written as follows:

$$\mathbf{C}^T \Phi(x_i, y_j) = f(x_i, y_j) + Q_1(x_i, y_j, s_{r_1}, t_{r_2}) + Q_2(x_i, y_j, s_{n-r_1}, t_{n-r_2}),$$

(4.7)
where:

\[ Q_1(x_i, y_j, s_{r_1}, t_{r_2}) = \sum_{r_2=0}^{n-r_1} \sum_{r_1=0}^{n-r_2} w_{r_1, r_2} k(x_i, y_j, s_{r_1}, t_{r_2}) C^T \Phi(s_{r_1}, t_{r_2}) \]

\[ Q_2(x_i, y_j, s_{n-r_2}, t_{n-r_1}) = \sum_{r_1=0}^{n} \sum_{r_2=0}^{n-r_1} w_{r_1, r_2} k(x_i, y_j, s_{n-r_2}, t_{n-r_1}) C^T \Phi(s_{n-r_2}, t_{n-r_1}) \]

\[ C^T \Phi(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij} \phi_{ij}(x, y). \]

This is a linear nonsingular system of equations, by solving this system, we obtain the unknown vector \( C^T \).

5 Numerical Example

In this section, a numerical example is solved to show the efficiency of the proposed method. The program has been provided by MAPLE.

**Example 5.1.** Consider the following two-dimensional linear Fredholm integral equation [7]:

\[ u(x, y) = g(x, y) + \int_0^1 \int_0^1 \frac{x}{(8+y)(1+t+s)} u(s, t) ds dt, \]

where \((x, y) \in [0, 1] \times [0, 1] \) and \( g(x, y) = \frac{1}{(1+x+y)^2} - \frac{x}{6(8+y)}. \) The exact solution is \( u(x, y) = \frac{1}{(1+x+y)^2}. \)

The errors of the RBFs with triangular method (RBF-TM) for \( n = 10 \) are shown in table 1 which are compared with 2D-TFs method mentioned in [7]. In this example, we consider \( \phi(r) = \frac{1}{1+r^2}. \)

In order to solve this integral equation, at first we calculate the weights \( w_i \) such as (3.2) to approximate the two-dimensional integral \( \int_0^1 \int_0^1 \frac{x}{(8+y)(1+t+s)} u(s, t) ds dt, \) then we apply the RBF-TM that is described in section 4 and solve the system (4.7) to find the unknown vector \( C^T \).

<table>
<thead>
<tr>
<th>nodes ((x, y))</th>
<th>Error of RBF-TM with ( n=10 )</th>
<th>Error of 2D-TFs method</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0.0,0.0))</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>((0.1,0.1))</td>
<td>3.6606e-006</td>
<td>7.5800e-003</td>
</tr>
<tr>
<td>((0.2,0.2))</td>
<td>7.2304e-006</td>
<td>9.5400e-003</td>
</tr>
<tr>
<td>((0.3,0.3))</td>
<td>1.07155e-005</td>
<td>9.6000e-003</td>
</tr>
<tr>
<td>((0.4,0.4))</td>
<td>1.41168e-005</td>
<td>9.0300e-003</td>
</tr>
<tr>
<td>((0.5,0.5))</td>
<td>1.74385e-005</td>
<td>3.0000e-004</td>
</tr>
<tr>
<td>((0.6,0.6))</td>
<td>2.06839e-005</td>
<td>4.3000e-004</td>
</tr>
<tr>
<td>((0.7,0.7))</td>
<td>2.38329e-005</td>
<td>2.4000e-003</td>
</tr>
<tr>
<td>((0.8,0.8))</td>
<td>2.69502e-005</td>
<td>2.6200e-003</td>
</tr>
<tr>
<td>((0.9,0.9))</td>
<td>2.99787e-005</td>
<td>2.8100e-003</td>
</tr>
<tr>
<td>((1.0,1.0))</td>
<td>3.29403e-005</td>
<td>-</td>
</tr>
</tbody>
</table>

6 Conclusion

In this paper, a new method based on the RBFs and triangular method was presented to solve the two-dimensional Fredholm integral equation. The proposed method is very simple and accurate to obtain the approximate solution of integral equation. Also, a new
and efficient quadrature method was suggested is called triangular method to approximate the two-dimensional integrals. The results of the example illustrated the importance of using this method.

References


