A new approach for ranking fuzzy numbers

M. Lotfi¹, S. Salahshour², F. Nasr Esfahani³, A. Jafarnejad¹

(1) Department of Management, University of Tehran, Iran
(2) Young Researchers and Elite Club, Mobarakeh Branch, Islamic Azad University, Mobarakeh, Iran
(3) Department of Knowledge Engineering and Decision Science, Mobarakeh Branch, Islamic Azad University, Mobarakeh, Iran

Abstract
In this paper, we propose a novel approach for constructing a preference fuzzy distance measure. For this purpose, we propose a new and effective preference ordering based on the Abbasbandy and Hajjari’s approach [2].

Keywords: Preference ordering; Metric properties.

1 Introduction

In order to rank fuzzy numbers, one fuzzy number needs to be evaluated and compared with the others, but this may not be easy. Jain [4,5] proposed a method using concept of maximizing set to order the fuzzy numbers in 1976. Some of these ranking methods have been compared and reviewed by Bortolan and Degani[6]. Chen and Hwang [29] proposed fuzzy multiple attribute decision making in 1992, Choobineh and Li[8] proposed an index for ordering fuzzy numbers in 1993. Also, Dias [9] ranked alternatives by ordering fuzzy numbers in 1993, Fortemps and Roubens [12] presented ranking and defuzzication methods based on area compensation in 1996. In 1998, Lee and Li [14] ranked fuzzy numbers based on two different criteria, namely, the fuzzy mean and the fuzzy spread of fuzzy numbers. So, Cheng [15] proposed the coefficient of variance (CV index) to improve Lee and Li’s ranking method. Chu and Tsao[24] pointed out that the shortcomings of Cheng’s method and suggested to rank fuzzy numbers with the area between the centroid point and the point of origin. Wang and co-workers [30] found that the centroid formulae provided by the paper[15, 24] are incorrect and have led to some misapplications. So, they presented the correct centroid formulae for the fuzzy numbers and justify them from the viewpoint of analytical geometry. Also, Chu’s method still has some drawbacks, which is expressed by Deng et. al [30] and carried out to modify by applying radius of gyration in 2006. However, Deng’s method also still has some drawbacks, i.e. it can not rank negative fuzzy numbers correctly (numerical example is illustrated in section 3). Notice that Deng’s method is independent from centroid point of fuzzy numbers[30]. Also, Abbasbandy et.al [1] proposed a ranking method based on sign distance in 2006, Asady et. al proposed an ordering approach by distance minimization[3]. Asady’s method has some drawbacks, i.e. for all triangular fuzzy numbers \( u = (x_0, \sigma, \beta) \) where \( x_0 = \frac{\sigma + \beta}{2} \) and also trapezoidal fuzzy number \( u = (x_0, y_0, \sigma, \beta) \) such that \( x_0 + y_0 = \frac{\sigma + \beta}{2} \) gives the same results. So, Abbasbandy and Hajjiari [2] proposed magnitude of fuzzy numbers in 2009 to improve Asady’s method. Also, Abbasbandy’s method has some drawbacks (numerical example is illustrated in section 3). These mentioned ranking methods are also placed in first or second class of

*Corresponding author. Email address: soheilsalahshour@yahoo.com
In this paper, we proposed a new ranking method for fuzzy numbers which is improved Abbasbandy’s approach[2]. Some algebraic properties of our proposed method are given, then is introduced an interval distance between two arbitrary fuzzy number.

2 Preliminaries

The basic definitions of a fuzzy number are given in [18, 19, 20] as follows:

**Definition 2.1.** A fuzzy number is a fuzzy set like \( u: \mathbb{R} \rightarrow [0, 1] \) which satisfies:

1. \( u \) is an upper semi-continuous function on,
2. \( u(x) = 0 \) outside some interval \([a,d]\),
3. There are real numbers \( a, b \) such as \( a \leq b \leq c \leq d \) and
   3.1 \( u(x) \) is a monotonic increasing function on \([a, b]\),
   3.2 \( u(x) \) is a monotonic decreasing function on \([c, d]\),
   3.3 \( u(x) = 1 \) for all \( x \in [b, c] \).

The membership function \( u \) is presented as

\[
\begin{align*}
  u(x) = \begin{cases} 
    u_L(x) & \text{if } x \in [a, b], \\ 
    1 & \text{if } x \in [b, c], \\ 
    u_R(x) & \text{if } x \in [c, d], \\ 
    0 & \text{otherwise}. 
  \end{cases}
\end{align*}
\]

(2.1)

where \( u_L: [a, b] \rightarrow [0, 1] \) and \( u_R: [c, d] \rightarrow [0, 1] \) are left and right membership functions of fuzzy number \( u \). Another definition for a fuzzy number is as follows:

**Definition 2.2.** A fuzzy number \( u \) in parametric form is a pair \((u, \overline{u})\) of functions \( u(r), \overline{u}(r), 0 \leq r \leq 1 \), which satisfy the following requirements:

1. \( u(r) \) is a bounded non-decreasing left continuous function in \([0, 1]\), and right continuous at 0,
2. \( \overline{u}(r) \) is a bounded non-increasing left continuous function in \([0, 1]\), and right continuous at 0,
3. \( u(r) \leq \overline{u}(r) \), \( 0 \leq r \leq 1 \).

The trapezoidal fuzzy number \( u = (x_0, y_0; \sigma, \beta) \), with two defuzzifier \( x_0, y_0 \), and left fuzziness \( \sigma > 0 \) and right fuzziness \( \beta > 0 \) is a fuzzy set where the membership function is as

\[
\begin{align*}
  u(x) = \begin{cases} 
    \frac{1}{\sigma} (x - x_0 + \sigma) & x_0 - \sigma \leq x \leq x_0, \\ 
    1 & x_0 \leq x \leq y_0, \\ 
    \frac{1}{\beta} (y_0 - x + \beta) & y_0 \leq x \leq y_0 + \beta, \\ 
    0 & \text{otherwise}. 
  \end{cases}
\end{align*}
\]

If \( x_0 = y_0 \), then \( u \) is called triangular fuzzy number and we write \( u = (x_0; \sigma, \beta) \). The support of fuzzy number \( u \) is defined as follows:

\[
\text{supp}(u) = \{ x | u(x) > 0 \},
\]

where \( \{ x | u(x) > 0 \} \) is closure of set \( \{ x | u(x) > 0 \} \).
3 New preference ordering for fuzzy numbers

For an arbitrary fuzzy number \( u = (x_0, y_0; \sigma, \beta) \), with parametric form \( u = (g(r), \pi(r)) \), we define the modified magnitude of the fuzzy number \( u \) as

\[
\text{MMag}(u, \lambda) = \frac{1}{2} \left( \int_0^r \{g(r) + \pi(r) + x_0 + y_0 + \sigma(\lambda - 1) + \beta \lambda \} f(r)dr \right)
\]

where the function \( f(r) \) is non-negative and increasing function on \([0, 1]\) with \( f(0) = 0, f(1) = 1 \) and \( \int_0^1 f(r)dr = \frac{1}{2} \) and \( \lambda \in [0, 1] \) is a decision maker parameter.

Obviously, modified magnitude of fuzzy number inherit most of properties of magnitude. So they have similar interpretation such that modified magnitude of fuzzy number \( u \) which is defined by (2), reflects the information on every membership degree and even have flexible behavior in compare of magnitude approach[2].

So, for two arbitrary fuzzy number \( u \) and \( v \) in \( E \), we define preference ranking of \( u \) and \( v \) by the MMag(.) on \( E \)(set of all fuzzy numbers) for all \( \lambda \in [0, 1] \) as follows:

1. \( \text{MMag}(u, \lambda) > \text{MMag}(v, \lambda) \) if and only if \( u \succ v \)
2. \( \text{MMag}(u, \lambda) < \text{MMag}(v, \lambda) \) if and only if \( u \prec v \)
3. \( \text{MMag}(u, \lambda) = \text{MMag}(v, \lambda) \) if and only if \( u \simeq v \).

Also we formulate the order \( \succeq \) and \( \preceq \) as \( u \succeq v \) if and only if \( u \succ v \) or \( u \simeq v \), \( u \preceq v \) if and only if \( u \prec v \) or \( u \simeq v \).

3.1 Some algebraic properties of \( \text{MMag}(., \lambda) \)

Here, we investigate the algebraic properties of modified magnitude of fuzzy number (2).

Property 3.1. Suppose that \( u = (x_0, y_0; \sigma_0, \beta_0) \) and \( v = (x_1, y_1; \sigma_1, \beta_1) \) are two arbitrary fuzzy numbers and \( \lambda \in [0, 1] \), then:

1. \( \text{MMag}(\alpha \odot u, \lambda) = \alpha \text{MMag}(u, \lambda) \),
2. \( \text{MMag}(u \oplus v, \lambda) = \text{MMag}(u, \lambda) + \text{MMag}(v, \lambda) \),

where \( \odot, \oplus \) are the fuzzy multiplication and addition operations, are defined by extension principle and \( \alpha \in \mathbb{R}^+ \).

Property 3.2. Suppose that \( u \) is a real number then

\[
\text{MMag}(u, \lambda) = u, \text{ for all } \lambda \in [0, 1]
\]

Property 3.3. For all symmetric fuzzy numbers \( u = (-x_0, x_0; \sigma, \sigma) \)

\[
\text{MMag}(u, \lambda) = \frac{2\sigma \lambda - 1}{4}, \text{ for all } \lambda \in [0, 1]
\]

Property 3.4. For two arbitrary symmetric fuzzy numbers \( u = (x_0, y_0; \sigma_0, \sigma_0) \) and \( v = (x_0, y_0; \sigma_1, \sigma_1) \)

\[
\text{MMag}(u, \lambda) - \text{MMag}(v, \lambda) = \frac{(2\lambda - 1)(\sigma_0 - \sigma_1)}{4}, \text{ for all } \lambda \in [0, 1]
\]

Consequently, Kerre’s reasonable properties are investigated for the ordering approach, see[21]

A-1 For an arbitrary finite subset \( \mathcal{A} \) of \( S \) and \( u \in \mathcal{A}; u \succeq u \).

A-2 For an arbitrary finite subset \( \mathcal{A} \) of \( S \) and \( (u, v) \in \mathcal{A}^2; u \succeq v \) and \( v \succeq u \) by MMag on \( \mathcal{A} \), this method should have \( u \sim v \).
A-3 For an arbitrary finite subset $\mathcal{A}$ of $S$ and $(u, v, w) \in \mathcal{A}^3$; $u \succeq v$ and $v \succeq w$ by MMag on $\mathcal{A}$, this method should have $u \succeq w$.

A-4 For an arbitrary finite subset $\mathcal{A}$ of S and $(u, v) \in \mathcal{A}^2$; $\inf \text{supp}(u) > \sup \text{supp}(v)$, this method should have $u \succeq v$.

A′-4 For an arbitrary finite subset $\mathcal{A}$ of $S$ and $(u, v) \in \mathcal{A}^2$; $\inf \text{supp}(u) > \sup \text{supp}(v)$, this method should have $u \succ v$.

A-5 Let $S, S'$ be two arbitrary finite sets of fuzzy quantities in which MMag can be applied and $u, v$ are in $S \cap S'$. This method obtain the ranking order $u \succeq v$ on $S'$ iff $u \succeq v$ on $S$.

A-6 Let $u, v, u + w$ and $v + w$ be elements of $S$. If $u \succeq v$ by MMag on $u, v$, then $u + w \succeq v + w$.

A′-6 Let $u, v, u + w$ and $v + w$ be elements of $S$. If $u \succ v$ by MMag on $u, v$, then $u + w \succ v + w$.

Property 3.5. Ranking approach (2), has the reasonable properties A1-A6.

3.2 Examples

Here, we take some illustrative examples to show the ability of proposed method.

Example 3.1. Let us consider two fuzzy numbers $A = (0; 1, 1)$ and $B = (1; 5, 1)$.

Intuitively, the ranking order is not $u \simeq v$. However by distance minimization [3], the ranking is $u \simeq v$. Also, by our method, $\text{MMag}(u, \lambda) = 0.5(\lambda - 0.5)$ and $\text{MMag}(v, \lambda) = 0.5(3\lambda - \frac{2}{5})$.

Example 3.2. Consider four fuzzy numbers $A = (1; 5, 1), B = \left(\frac{1}{2}; 2, 1\right), C = (2; 9, 1)$ and $D = (0, 1; 2, 0)$.

Intuitively, the ranking order is not $B \simeq D \simeq A \simeq C$. However by distance minimization the ranking order is $B \simeq D \simeq A \simeq C$. By applying our proposed method $\text{MMag}(A, \lambda) = 0.5(3\lambda - \frac{2}{5}), \text{MMag}(B, \lambda) = 0.5(1.5\lambda - 0.6), \text{MMag}(C, \lambda) = 0.5(5\lambda - \frac{11}{4})$ and $\text{MMag}(D, \lambda) = 0.5(\lambda - 0.333)$. For pessimistic case ($\lambda = 0$) ordering is $C < B < A < D$ and for optimistic case ($\lambda = 1$)is $B < D < A < C$ (see Fig.2).

Example 3.3. Consider the following sets [25]

Set1: $A = (0.5; 0.1, 0.5), B = (0.7; 0.3, 0.2), C = (0.9; 0.5, 0.1)$.
Set2: $A = (0.4, 0.7; 0.1, 0.2)$ (trapezoidal fuzzy number), $B = (0.7; 0.4, 0.2), C = (0.7; 0.2, 0.2)$.
Set3: $A = (0.5, 0.2, 0.2), B = (0.5, 0.8; 0.2, 0.1)$ (trapezoidal fuzzy number), $C = (0.5, 0.2, 0.4)$.
Set4: $A = (0.4, 0.7; 0.4, 0.1)$ (trapezoidal fuzzy number), $B = (0.5, 0.3, 0.4), C = (0.6; 0.5, 0.2)$.

For more detail see Table1.
### Table 1: Comparative results of example 3.3

<table>
<thead>
<tr>
<th>Authors</th>
<th>FNs</th>
<th>Set1</th>
<th>Set2</th>
<th>Set3</th>
<th>Set4</th>
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<tbody>
<tr>
<td>MMag($\lambda$)</td>
<td>A</td>
<td>0.5083+.15$\lambda$</td>
<td>0.5334+.075$\lambda$</td>
<td>0.45+.1$\lambda$</td>
<td>0.425+.125$\lambda$</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0.625+.15$\lambda$</td>
<td>0.5833+.15$\lambda$</td>
<td>0.5916+.075$\lambda$</td>
<td>0.433+.175$\lambda$</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>0.7416+.15$\lambda$</td>
<td>0.65+.1$\lambda$</td>
<td>0.4666+.15$\lambda$</td>
<td>0.45+.175$\lambda$</td>
</tr>
<tr>
<td>Results($\lambda=0$)</td>
<td>A ≺ B ≺ C</td>
<td>A ≺ B ≺ C</td>
<td>A ≺ C ≺ B</td>
<td>A ≺ B ≺ C</td>
<td></td>
</tr>
<tr>
<td>Results($\lambda=1$)</td>
<td>A ≺ B ≺ C</td>
<td>A ≺ B ≺ C</td>
<td>A ≺ C ≺ B</td>
<td>A ≺ B ≺ C</td>
<td></td>
</tr>
<tr>
<td>Mag(.)</td>
<td>A</td>
<td>0.5334</td>
<td>0.5584</td>
<td>0.5000</td>
<td>0.5250</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0.7000</td>
<td>0.6334</td>
<td>0.6416</td>
<td>0.5084</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>0.8666</td>
<td>0.7000</td>
<td>0.5166</td>
<td>0.5750</td>
</tr>
<tr>
<td>Results</td>
<td>A ≺ B ≺ C</td>
<td>A ≺ B ≺ C</td>
<td>A ≺ C ≺ B</td>
<td>B ≺ A ≺ C</td>
<td></td>
</tr>
<tr>
<td>Distance Minimization</td>
<td>A</td>
<td>0.6</td>
<td>0.575</td>
<td>0.5</td>
<td>0.475</td>
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<tr>
<td></td>
<td>B</td>
<td>0.7</td>
<td>0.65</td>
<td>0.625</td>
<td>0.525</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>0.9</td>
<td>0.7</td>
<td>0.55</td>
<td>0.525</td>
</tr>
<tr>
<td>Results</td>
<td>A ≺ B ≺ C</td>
<td>A ≺ B ≺ C</td>
<td>A ≺ C ≺ B</td>
<td>A ≺ B ≺ C</td>
<td></td>
</tr>
<tr>
<td>Choobineh and Li</td>
<td>A</td>
<td>0.3333</td>
<td>0.5480</td>
<td>0.3330</td>
<td>0.5000</td>
</tr>
<tr>
<td></td>
<td>B</td>
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<td>0.5833</td>
</tr>
<tr>
<td></td>
<td>C</td>
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<td>0.6670</td>
<td>0.5417</td>
<td>0.6111</td>
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<tr>
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<td>A ≺ C ≺ B</td>
<td>A ≺ B ≺ C</td>
<td></td>
</tr>
<tr>
<td>Yager</td>
<td>A</td>
<td>0.6000</td>
<td>0.5750</td>
<td>0.5000</td>
<td>0.4500</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0.7000</td>
<td>0.6500</td>
<td>0.5500</td>
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<tr>
<td></td>
<td>C</td>
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<td>0.7000</td>
<td>0.6250</td>
<td>0.5500</td>
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<td>A ≺ B ≺ C</td>
<td>A ≺ C ≺ B</td>
<td>A ≺ B ≺ C</td>
<td></td>
</tr>
<tr>
<td>Chen</td>
<td>A</td>
<td>0.3375</td>
<td>0.4315</td>
<td>0.3750</td>
<td>0.5200</td>
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<td></td>
<td>B</td>
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<td>0.5625</td>
<td>0.4250</td>
<td>0.5700</td>
</tr>
<tr>
<td></td>
<td>C</td>
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<tr>
<td>Results</td>
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<td>A ≺ B ≺ C</td>
<td>A ≺ B ≺ C</td>
<td>A ≺ B ≺ C</td>
<td></td>
</tr>
<tr>
<td>Baldwin and Guild</td>
<td>A</td>
<td>0.3000</td>
<td>0.2700</td>
<td>0.2700</td>
<td>0.4000</td>
</tr>
<tr>
<td></td>
<td>B</td>
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<td>0.2700</td>
<td>0.3700</td>
<td>0.4200</td>
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<td>0.4400</td>
<td>0.3700</td>
<td>0.4500</td>
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<td>A ≺ B ≺ C</td>
<td>A ≺ B ≺ C</td>
<td>A ≺ B ≺ C</td>
<td></td>
</tr>
</tbody>
</table>

**Example 3.4.** Consider three fuzzy number $A = (6;1,1), B = (6;0.1,1)$ and $C = (6,0,1)$.

Using proposed method, we get the following:

- $\text{MMag}(A, \lambda) = 0.5(11.5 + \lambda)$
- $\text{MMag}(B, \lambda) = 0.5(12.1 + 0.55\lambda)$
- $\text{MMag}(C, \lambda) = 0.5(12.17 + 0.5\lambda)$
References

http://dx.doi.org/10.1016/j.ins.2005.03.013

http://dx.doi.org/10.1016/j.camwa.2008.10.090

http://dx.doi.org/10.1016/j.apm.2006.10.018

http://dx.doi.org/10.1109/TSMC.1976.4309421

http://dx.doi.org/10.1080/002077277018

http://dx.doi.org/10.1016/0165-0114(85)90012-0

http://dx.doi.org/10.1007/978-3-642-46768-4

http://dx.doi.org/10.1016/0165-0114(93)90374-Q

http://dx.doi.org/10.1016/0165-0114(93)90151-7

http://dx.doi.org/10.1016/0165-0114(94)90002-7

http://dx.doi.org/10.1016/0165-0114(94)90153-8

http://dx.doi.org/10.1016/0165-0114(95)00273-1

http://dx.doi.org/10.1016/S0165-0114(97)00243-1

http://dx.doi.org/10.1016/0898-1221(88)90124-1

http://dx.doi.org/10.1016/S0165-0114(96)00272-2

http://dx.doi.org/10.1016/0165-0114(87)90028-5

http://dx.doi.org/10.1016/0165-0114(92)90062-9


http://dx.doi.org/10.1007/978-94-015-7949-0

http://dx.doi.org/10.1016/S0165-0114(99)00063-9

http://dx.doi.org/10.1016/0165-0114(79)90028-9

http://dx.doi.org/10.1016/0165-0114(85)90050-8

http://dx.doi.org/10.1016/S0898-1221(01)00277-2

http://dx.doi.org/10.1016/S0165-0114(98)00122-5


http://dx.doi.org/10.1016/S0019-9958(65)90241-X

http://dx.doi.org/10.1007/978-3-642-46768-4

http://dx.doi.org/10.1016/0005-1098(77)90008-5

http://dx.doi.org/10.1016/j.fss.2005.11.006

http://dx.doi.org/10.1016/j.camwa.2004.11.022

http://dx.doi.org/10.1016/S0165-0114(01)00195-6
http://dx.doi.org/10.1016/j.mcm.2005.09.025

http://dx.doi.org/10.1016/S0165-0114(97)00090-0