Convergence and stability properties Euler method for solving fuzzy Stochastic differential equations

A. Seyed Alavi Nobar*

Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.

Abstract
In this paper we propose convergence and stability properties of Euler method for solving fuzzy stochastic differential equations under generalized differentiability concept. It is used to find the analytical solution of method for some fuzzy stochastic differential equations (FSDEs). The related theorems and properties are proved in detail and the method is illustrated by solving some examples.

Keywords: Fuzzy Stochastic differential equations, Generalized differentiability, Fuzzy modified Euler method.

1 Introduction

Fuzzy stochastic differential equations (FSDEs) deal with the real phenomena not only with randomness but also with fuzziness. Puri and Ralescu introduced fuzzy set-valued random variable in [8]. In the literature, it contains different definitions of fuzzy random variables. Kwakernaak has proposed the concept of fuzzy random variable for the first time [15]. Further, it was used by Kruse and Meyer [16]. In [8], there appear two notions of measurability of fuzzy mappings. The relations between different concepts of measurability for fuzzy random variables are contained in the papers of Colubi et al. [18], Terán Agraz [17], López-Díaz and Ralescu [19].

Marek T. Malinowski considered the fuzzy stochastic differential equations (FSDEs) under suitable conditions the Peano type theorem on existence of solutions and show their application by an example [3]. In [3, 4, 5, 6], the authour consider the fuzzy random differential equation with initial value.

\[
\begin{align*}
&x'(t, \omega) \mid_{[a,b]} \overset{P} \equiv f_\omega(t, x(t, \omega)), \\
x(t_0, \omega) \overset{P} = x_0(\omega) \in \mathbb{E},
\end{align*}
\]

where \( f : [a, b] \times \Omega \times \mathbb{E} \to \mathbb{E} \) and the symbol ' denotes the fuzzy derivative is understood in the sense of Puri and Ralescu [8]. The existence and uniqueness of a Cauchy problem is then obtained under an assumption that the coefficients satisfy a condition with the Lipschitz constant. Marek T. Malinowski showed that if \( f \) is continuous and \( f_\omega(t, x) \) satisfies the Lipschitz condition with respect to \( x \), then there exists a unique local solution for the fuzzy random initial value problem (1.1). In [20] the existence and uniqueness of the solution for RFDEs with non-Lipschitz coefficients is proven.

Inspired and motivated by T. Allahviranloo et.al [9, 10, 12, 13] and Marek T. Malinowski [3, 4, 5, 6], we consider the
existence and uniqueness results for random fuzzy differential equations. For the existence and uniqueness, we use the method of successive approximations. Some kind of boundedness of solution is established. We provide examples to illustrate the results.

2 Preliminaries

The basic definition of fuzzy numbers is given in [21]. We denote the set of all real numbers by \( \mathbb{R} \) and the set of all fuzzy number on \( \mathbb{R} \) is indicated by \( \mathcal{E} \). A fuzzy number is a mapping \( u : \mathbb{R} \rightarrow [0, 1] \) with the following properties:

(a) \( u \) is upper semi-continuous,
(b) \( u \) is fuzzy convex, i.e., \( u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\} \) for all \( x, y \in \mathbb{R}, \lambda \in [0, 1] \),
(c) \( u \) is normal, i.e., \( \exists x_0 \in \mathbb{R} \) for which \( u(x_0) = 1 \),
(d) \( \text{supp} u = \{ x \in \mathbb{R} \mid u(x) > 0 \} \) is the support of the \( u \), and its closure \( \text{cl}(\text{supp} u) \) is compact.
An equivalent parametric definition is also given in [7, 11, 22] as follows:

**Definition 2.1.** A fuzzy number \( u \) in parametric form is a pair \( (u, \Pi) \) of functions \( u(r), \Pi(r) \), \( 0 \leq r \leq 1 \), which satisfy the following requirements:

1. \( u(r) \) is a bounded non-decreasing left continuous function in \( [0, 1] \), and right continuous at 0,
2. \( \Pi(r) \) is a bounded non-increasing left continuous function in \( [0, 1] \), and right continuous at 0,
3. \( u(r) \leq \Pi(r) \), \( 0 \leq r \leq 1 \).

Moreover, we also can present the \( r \)-cut representation of fuzzy number as \([u]^r = [u(r), \Pi(r)]\) for all \( 0 \leq r \leq 1 \). According to Zadeh’s extension principle, operation of addition on \( \mathcal{E} \) is defined by

\[
(u \oplus v)(x) = \sup_{y \in \mathbb{R}} \{u(y), v(x-y)\}, \quad x \in \mathbb{R},
\]

and scalar multiplication of a fuzzy number is given by

\[
(k \odot u)(x) = \begin{cases} u(x/k), & k > 0, \\ 0, & k = 0, \end{cases}
\]

where \( \hat{0} \in \mathcal{E} \).

The Hausdorff distance between fuzzy numbers given by \( d : \mathcal{E} \times \mathcal{E} \rightarrow \mathbb{R} \cup \{0\} \),

\[
d(u, v) = \sup_{r \in [0,1]} \max\{|u(r) - v(r)|, |\Pi(r) - \Pi(\alpha)|\},
\]

where \( u = (u(r), \Pi(r)), v = (v(r), \Pi(\alpha)) \subset \mathbb{R} \) is utilized in [1]. Then, it is easy to see that \( d \) is a metric in \( \mathcal{E} \) and has the following properties (see [8]):

1. \( d(u + w, v + w) = d(u, v) \), \( \forall u, v, w \in \mathcal{E} \),
2. \( d(\alpha u, \alpha v) = |\alpha| d(u, v) \), \( \forall \alpha \in \mathbb{R}, u, v \in \mathcal{E} \),
3. \( d(u + v, w + e) \leq d(u, w) + d(v, e) \), \( \forall u, v, w, e \in \mathcal{E} \),
4. \( (d, \mathcal{E}) \) is a complete metric space.

**Theorem 2.1.** [14]. Let \( f(x) \) be a fuzzy-valued function on \([a, \infty)\) and it is represented by \( (f(x, r), \overline{f}(x, r)) \). For any fixed \( r \in [0, 1] \), assume \( f(x, r) \) and \( \overline{f}(x, r) \) are Riemann-integrable on \([a, b]\) for every \( b \geq a \), and assume there are two positive \( M(r) \) and \( \overline{M}(r) \) such that \( \int_a^b |f(x, r)| \, dx \leq M(r) \) and \( \int_a^b |\overline{f}(x, r)| \, dx \leq \overline{M}(r) \) for every \( b \geq a \). Then \( f(x) \) is improper fuzzy Riemann-integrable on \([a, \infty)\) and the improper fuzzy Riemann-integral is a fuzzy number. Furthermore, we have:

\[
\int_a^\infty f(x) \, dx = \left( \int_a^\infty f(x, r) \, dx, \int_a^\infty \overline{f}(x, r) \, dx \right).
\]
Definition 2.2. Let \( x, y \in \mathbb{E} \). If there exists \( z \in \mathbb{E} \) such that \( x = y + z \), then \( z \) is called the H-difference of \( x \) and \( y \), and it is denoted by \( x \ominus y \).

In this paper, the sign ”\( \ominus \)” always stands for H-difference, and also note that \( x \ominus y \neq x + (-1)y \).

3 Fuzzy stochastic initial value problem

This section begins by reviewing some important concepts, definitions and results related to the fuzzy random calculus that will play an important role in the understanding and results of the later sections. Let \((\Omega, \mathcal{F}, P)\) be complete probability space. A function \( X_0 : \Omega \to \mathbb{E} \) is called fuzzy random variable, if for all \( \alpha \in [0, 1] \) the set-valued mapping \([X_0(\cdot)]^\alpha : \Omega \to \mathbb{E}\) is a measurable, i.e. for every closed set \( C \in \mathbb{R} \)

\[ \{ \omega \in \Omega \mid [X_0(\omega)]^\alpha \cap C \neq \emptyset \} \in \mathcal{F} \]

A mapping \( X : [a, b] \times \Omega \to \mathbb{E} \) is said to be a fuzzy stochastic process if \( X(\cdot, \omega) \) is a fuzzy set-valued function with any fixed \( \omega \in \Omega \). Assume that \( f : \Omega \times [a, b] \times \mathbb{E} \to \mathbb{E} \) satisfies:

\( f(1) \): \( f(t, u) : \Omega \to \mathbb{E} \) is a fuzzy random variable for every \( t \in [a, b] \), and every \( u \in \mathbb{E} \); \n
\( f(2) \): with probability 1 \((P.1)\) the mapping \( f_0(\cdot, \cdot) : [a, b] \times \mathbb{E} \to \mathbb{E} \) is a continuous fuzzy mapping at every point \((x, u) \in [a, b] \times \mathbb{E}\). i.e. there exists \( \Omega_0 \in \Omega \) with \( P(\Omega_0) = 1 \) such that for every \( \omega \in \Omega_0 \) the following is true:

for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for every \( t \in [a, b] \) and every \( u \in \mathbb{E} \) it holds

\[ \max \{|t_0 - t|, d(u, u_0)\} < \delta \implies d(f_0(t, u), f_0(t_0, u_0)) < \varepsilon \]

If there exists \( \Omega^* \subset \Omega \) such that \( P(\Omega^*) = 1 \) and for every fixed \( \omega \in \Omega^* \) it holds \( X(t, \omega) = Y(t, \omega) \) for every \( t \in [a, b] \), where \( X, Y \) are stochastic processes, then we will write \( X(t, \omega) = Y(t, \omega) \) for every \( t \in [0, 1] \) with \( P.1 \).

Here, we will consider an stochastic fuzzy initial value problem for every \( t \in [a, b] \) with \( P.1 \) as follows:

\[
\begin{aligned}
X'(t, \omega) &= f_0(t, X(t, \omega)), \\
X(a, \omega)(0) &= X_0(\omega).
\end{aligned}
\] (3.3)

Therefore we have the following definition of the solution to (2.2).

Definition 3.1. A solution \( X \) is unique if for every \( t \in [a, b] \) with \( P.1 \) it holds,

\[ d(X(t, \omega), Y(t, \omega)) = 0, \]

for any fuzzy stochastic process \( Y : [a, b] \times \Omega \to \mathbb{E} \) which is a solution of (2.2).

Theorem 3.1. Let \( f : \Omega \times [a, b] \times \mathbb{E} \to \mathbb{E} \) and consider \( a = t_0 < t_1 < \ldots < t_n = b \) be a division of the interval \([a, b]\) such that \( f \) is (i) or (ii)-differentiable in Definition 2.2 on end of the intervals \([t_{i-1}, t_i], i = 1, 2, 3, \ldots, n\) with the same kind of differentiability on each subinterval. Then we have:

\[
\int_a^b f_0(s, X(s, \omega))ds = \sum_{i=1}^{n} (f_0(t_i, X(t_i, \omega)) - f_0(t_{i-1}, X(t_{i-1}, \omega))) \oplus (-1) \oplus \sum_{k=1}^{J} (f_0(t_{k-1}, X(t_{k-1}, \omega)) - f_0(t_k, X(t_k, \omega))),
\] (3.4)

where \( I = \{i \mid i \in \{1, 2, 3, \ldots, n\}\} \), such that \( f_0 \) is (i)-differentiable on \((t_{i-1}, t_i)\), and \( J = \{k \mid k \in \{1, 2, 3, \ldots, n\}\} \), such that \( f_0 \) is (ii)-differentiable on \((t_{k-1}, t_k)\).

Proof. See Theorem 18 in [1].
Lemma 3.1. Let \( f : \Omega \times [a, b] \times \mathbb{E} \to \mathbb{E} \) be a fuzzy stochastic process where \( f \) is supported to be continuous with P.1 then for \( X_0 : \Omega \to \mathbb{E} \), the fuzzy stochastic differential equation (1.1) is equivalent to one of the integral equations: \( X(t, \omega) = X_0(\omega) \oplus \int_a^t f_\omega(s, X(s, \omega)) ds \), for every \( t \in [0, 1] \) with P.1 on some interval \((a, b) \subseteq \mathbb{R}\), depending on the strongly differentiability considered. Definition 2.1 or Definition 2.2, respectively. Here the equivalence between two equations means that any solution of an equation is a solution too for the other one.

**Proof.** It is immediately by Theorem 3.2.

**Remark 3.1.** In the case of strongly generalized differentiability, for the \( X(t, \omega) = f_\omega(t, X(t(\omega))) \), we may attach two different integral equations in which the second solution can be written in the form \( X(t, \omega) = X_0(\omega) - (-1) \oplus \int_a^t f_\omega(s, X(s, \omega)) ds \).

**Theorem 3.2.** Let \( f : \Omega \times [a, b] \times \mathbb{E} \to \mathbb{E} \), satisfies \( f(1) - f(2) \) and assume that there exists a stochastic process \( L : [a, b] \times \Omega \to \mathbb{R}_+, L(\cdot, \omega) \) continuous with P.1, such that
\[
d(f_\omega(t, u), f_\omega(t, v)) \leq L(t, \omega)d(u, v)
\]
for every \( t \in [a, b] \) and every \( u, v \in \mathbb{E} \) with P.1. Let \( X_0 : \Omega \to \mathbb{E} \) be a fuzzy random variable such that for every \( t \in [a, b] \)
and every \( \omega \in \Omega \), the following successive iterations converge to these two solutions, respectively,
\[
\begin{align*}
X_n+1(t, \omega) &= X_0(\omega) \oplus \int_a^t f_\omega(s, X_n(s, \omega)) ds, n \in \mathbb{N} \\
X_0(t, \omega) &= X_0(\omega).
\end{align*}
\]
(3.5)

\[
\begin{align*}
Y_{n+1}(t, \omega) &= Y_0(\omega) - (-1) \oplus \int_a^t f_\omega(s, Y_n(s, \omega)) ds, n \in \mathbb{N} \\
Y_0(t, \omega) &= Y_0(\omega).
\end{align*}
\]
(3.6)

where \( X_0(\omega) = Y_0(\omega) \).

**Proof.** For every \( t \in [a, b] \) with P.1, we have
\[
d(X_1(t, \omega), X_0(t, \omega)) = d(\int_a^t f_\omega(s, X_0(s, \omega)) ds, \tilde{0}) \leq \int_a^t d(f_\omega(s, X_0(s, \omega)), \tilde{0}) ds \leq M(t - a) \leq M(b - a)
\]
(3.7)

and for every \( n \geq 1 \),
\[
d(X_{n+1}(t, \omega), X_n(t, \omega)) = d(\int_a^t f_\omega(s, X_n(s, \omega)) ds, \int_a^t f_\omega(s, X_{n-1}(s, \omega)) ds)
\]
\[
\leq \int_a^t d(f_\omega(s, X_n(s, \omega)), f_\omega(s, X_{n-1}(s, \omega))) ds
\]
\[
\leq \tilde{L}(\omega) \int_a^t d(X_n(s, \omega), X_{n-1}(s, \omega)) ds.
\]

where \( \tilde{L}(\omega) = \sup_{t \in [a, b]} L(t, \omega) \).

Thus, by successive substitution by (3.7), we have
\[
d(X_{n+1}(t, \omega), X_n(t, \omega)) \leq M\tilde{L}^n(\omega) \frac{(t - a)^{n+1}}{(n+1)!} \leq M\tilde{L}^n(\omega) \frac{(b - a)^{n+1}}{(n+1)!}
\]
(3.8)

\[\square\]
Observe that for every $n \in \mathbb{N} \cup \{0\}$ the functions $X_n(\cdot, \omega) : [a, b] \to \mathbb{E}$ are continuous with P.1. Indeed, $X_0(t, \omega)$ does not depend on $t$, and for the right-sided continuity of $X_1(\cdot, \omega)$ we have

$$d(X_1(t + h, \omega), X_1(t, \omega)) = d\left(\int_a^{t+h} f_0(s, X_0(\omega)) ds, \int_a^t f_0(s, X_0(\omega)) ds\right)$$

$$= d\left(\int_t^{t+h} f_0(s, X_0(\omega)) ds, 0\right) \leq \left(\int_t^{t+h} d(f_0(s, X_0(\omega)), 0) ds\right) \leq Mh \to 0$$

as $h \searrow 0, \forall t \in [a, b]$ with P.1

Similarly for the left-sided continuity one has

$$d(X_1(t - h, \omega), X_1(t, \omega)) \leq Mh \to 0$$

as $h \searrow 0, \forall t \in (a, b]$ with P.1

For every $n \geq 2$ by using (3.8), we have

$$d(X_n(t + h, \omega), X_n(t, \omega)) = d\left(\int_a^{t+h} f_0(s, X_{n-1}(s, \omega)) ds, \int_a^t f_0(s, X_{n-1}(s, \omega)) ds\right)$$

$$= d\left(\int_t^{t+h} f_0(s, X_{n-1}(s, \omega)) ds, 0\right) \leq \left(\int_t^{t+h} d(f_0(s, X_{n-1}(s, \omega)), 0) ds\right)$$

$$\leq \int_t^{t+h} \left(d(f_0(s, X_0(\omega)), 0) + \sum_{k=1}^{n-1} d(f_0(s, X_k(s, \omega)), f_0(s, X_{k-1}(s, \omega)))\right) ds$$

$$\leq \left(\int_t^{t+h} (M + \bar{L}(\omega) \sum_{k=1}^{n-1} M\bar{L}^{k-1}(\omega) \frac{(b-a)^k}{k!}) ds\right)$$

$$= Mh(1 + \sum_{k=1}^{n-1} \frac{\bar{L}^k(\omega)(b-a)^k}{k!}).$$

when $h \searrow 0$, for every $t \in [a, b]$ with P.1. For last term $\searrow 0$ which means almost surely (with respect to the measure $P$) convergence.

Similar inequalities are obtained for $d(X_n(t - h, \omega), X_n(t, \omega))$ Hence appropriate right-sided and left-sided continuity with P.1 of the functions $X_n(\cdot, \omega), n \geq 2$, follows. Now we shall show that $\{X_{n+1}(\cdot, \omega)\}$ is (i)-differentiable and $X_{n+1}(t, \omega) = f_0(t, X_n(t, \omega))$ for every $t \in [a, b]$ with P.1.

For every $n \geq 1$ by using (3.5).

$$X_{n+1}(t, \omega) \oplus \int_t^{t+h} f_0(s, X_n(s, \omega)) ds = X_0(\omega) \oplus \int_a^t f_0(s, X_n(s, \omega)) ds \oplus \int_t^{t+h} f_0(s, X_n(s, \omega)) ds =$$

$$= X_0(\omega) \oplus \int_a^{t+h} f_0(s, X_n(s, \omega)) ds = X_{n+1}(t + h, \omega),$$

therefore,

$$X_{n+1}(t + h, \omega) - X_{n+1}(t, \omega) = \int_t^{t+h} f_0(s, X_n(s, \omega)) ds. \quad (3.10)$$

Multiplying both sides of (3.6) by $\frac{1}{h}$ and limiting as $h \searrow 0$ we have by Definition 2.1

$$\lim_{h \searrow 0} \frac{X_{n+1}(t + h, \omega) - X_{n+1}(t, \omega)}{h}$$
Also, for every $t \in [a, b]$ with $P1$, 
\[ d\left(\frac{1}{h} \cap \int_t^{t+h} f_0(s, X_n(s, \omega))ds \right) = d\left(\frac{1}{h} \cap \int_t^{t+h} f_0(s, X_n(s, \omega))ds, f_0(t, X_n(t, \omega)) \right) \]
\[ = \frac{1}{h} \cap \int_t^{t+h} f_0(s, X_n(s, \omega))ds, \frac{1}{h} \cap \int_t^{t+h} f_0(t, X_n(t, \omega))ds \]
\[ \leq \sup_{|x-t| \leq h} d\left(f_0(s, X_n(s, \omega)), f_0(t, X_n(t, \omega)) \right) \]
\[ \leq \tilde{L}(\omega) \sup_{|x-t| \leq h} d(X_n(s, \omega), X_n(t, \omega)), \]

and since $X_n(\cdot, \omega)$ is continuous, for $h \downarrow 0$ the last term $\downarrow 0$ which means that
\[ \lim_{h \downarrow 0} \frac{X_{n+1}(t+h, \omega) - X_{n+1}(t, \omega)}{h} = f_0(t, X_n(t, \omega)) . \]

(3.11)

thereby Similar to (3.9) we can obtain
\[ X_{n+1}(t, \omega) - X_{n+1}(t-h, \omega) = \int_{t-h}^t f_0(s, X_n(s, \omega))ds, \]

which similar to (3.10) we get
\[ \lim_{h \downarrow 0} \frac{X_{n+1}(t, \omega) - X_{n+1}(t-h, \omega)}{h} = f_0(t, X_n(t, \omega)) . \]

(3.12)

Thereby $X_n(\cdot, \omega)$ is (i)-differentiable according to Definition 2.1 and for every $t \in [a, b]$, we have
\[ X'_{n+1}(t, \omega) = f_0(t, X_n(t, \omega)). \]

(3.13)

In the sequel we shall show that for the sequence $X_n(t, \omega)$ the cauchy convergence condition is saitsed uniformly on the variable $t$ with $P1$, and as a consequence $X_n(\cdot, \omega)$ is uniformly convergent with $P1$.

For $n > m > 0$ with $P1$. using (3.8) one obtains
\[ \sup_{t \in [a, b]} d\left(X_n(t, \omega), X_m(t, \omega) \right) \leq \sum_{k=m}^{n-1} \sup_{t \in [a, b]} d\left(X_{k+1}(t, \omega), X_k(t, \omega) \right) \]
\[ \leq M \sum_{k=m}^{n-1} \tilde{L}^k(\omega) \frac{(b-a)^{k+1}}{(k+1)!} . \]

The almost surely convergence of the series $\sum_{n=1}^{\infty} \tilde{L}^n(\omega) \frac{(b-a)^n}{(n)!}$ implies that for any $\varepsilon > 0$ one can find $n_0 > N$, large enough, such that for $n, m_0$ with $P1$
\[ \sup_{t \in [a, b]} d\left(X_n(t, \omega), X_m(t, \omega) \right) \leq \varepsilon \]

(3.14)

$(R_F, d)$ is a complete metric space and (3.14) holds. therefore, the sequence $X_n(\cdot, \omega)$ is uniformly convergence to $X(\cdot, \omega)$ for every $t \in [a, b]$ with $P1$. 

International Scientific Publications and Consulting Services
Note that for every $t \in [a, b]$ the functions $X_t : \Omega \to \mathcal{R}_F$ dened by (2.2) are fuzzy random variables. Also, $X_n(\cdot, \cdot)$ is a sequence of continuous fuzzy stochastic processes. Hence $X(\cdot, \cdot)$ is a solution of the stochastic fuzzy initial value problem (1.1).

Since $f_\omega$ is continuous, using (3.8) for every $t \in [a, b]$ with $P$, as $n \to \infty$ we have

$$X'(t, \omega) = f_\omega(t, X(t, \omega))$$

(3.15)

Therefore $X(t, \omega)$ is a solution of the stochastic fuzzy initial value problem (1.1) for every $t \in [a, b]$ with $P$.

Let us denote the $\alpha$-cuts of $X(t, \omega)$ by the level sets $[X(t, \omega)]^\alpha = [X(t, \omega; \alpha), X(t, \omega; \alpha)]$, $\alpha \in (0, 1]$ then, we can rewrite the $\alpha$-cuts of the relation (2.2) as the form

$$\begin{align*}
X_{n+1}(t, \omega; \alpha) &= X_0(\omega; \alpha) \oplus \int_a^t f_\omega(s, X_n(s, \omega; \alpha))ds, \\
X_{n+1}(t, \omega; \alpha) &= X_0(\omega; \alpha) \oplus \int_a^t f_\omega(s, X_n(s, \omega; \alpha))ds,
\end{align*}$$

(3.16)

with initial conditions $X(0, \omega; \alpha) = X_0(\omega; \alpha)$, $X(0, \omega; \alpha) = X_0(\omega; \alpha)$ Similar to the above case (relations (3.7), (3.8)), for the (ii)-differentiability we have by (3.6) for every $t \in [a, b]$ with $P$

$$d(Y_1(t, \omega), Y_0(t, \omega)) \leq M(b - a)$$

(3.17)

and for every $n \geq 1$

$$d(Y_{n+1}(t, \omega), Y_n(t, \omega)) \leq \bar{L}(\omega) \int_a^t d(Y_n(s, \omega), Y_{n-1}(s, \omega))ds \leq M\bar{L}^n(\omega) \frac{(b - a)^n}{(n + 1)!}$$

(3.18)

Similar to the above case, observe that for every $n \in N \cup \{0\}$ the functions $Y_n(\cdot, \cdot) : [a, b] \to \mathcal{R}_F$ are continuous with $P$.

Now we shall show that $Y_{n+1}(\cdot, \cdot)$ is (ii)-differentiable and $Y'_{n+1}(t, \omega) = f_\omega(t, Y_n(t, \omega))$ for every $t \in [a, b]$ with $P$.

For every $n \geq 1$, we have

$$\begin{align*}
Y_{n+1}(t + h, \omega) &\ominus (1) \ominus \int_t^{t+h} f_\omega(s, Y_n(s, \omega))ds \\
&= Y_0(\omega) - (1) \ominus \int_t^{t+h} f_\omega(s, Y_n(s, \omega))ds \ominus (1) \ominus \int_t^{t+h} f_\omega(s, Y_n(s, \omega))ds \\
&= Y_0(t) - (1) \ominus \int_t^{t+h} f_\omega(s, Y_n(s, \omega))ds \ominus (1) \ominus \int_t^{t+h} f_\omega(s, Y_n(s, \omega))ds \\
&= (1) \ominus \int_t^{t+h} f_\omega(s, Y_n(s, \omega))ds = Y_0(t) - (1) \ominus \int_t^{t+h} f_\omega(s, Y_n(s, \omega))ds \\
&= Y_{n+1}(t, \omega)
\end{align*}$$

therefore,

$$Y_{n+1}(t, \omega) - Y_{n+1}(t + h, \omega) = (1) \ominus \int_t^{t+h} f_\omega(s, Y_n(s, \omega))ds$$

(3.19)

Multiplying both sides of (3.19) by $\frac{1}{h}$ and limiting as $h \searrow 0$ we have by Definition 2.2,

$$\lim_{h \searrow 0} \frac{Y_{n+1}(t, \omega) - Y_{n+1}(t + h, \omega)}{-h}$$

$$\lim_{h \searrow 0} \frac{1}{h} \int_t^{t+h} f_\omega(s, Y_n(s, \omega))ds.$$

We observe that

$$d\left(\frac{1}{h} \int_t^{t+h} f_\omega(s, Y_n(s, \omega))ds, f_\omega(t, Y_n(t, \omega))\right)$$
which similar to (3.20) we get
\[
(FSDEs) \text{ driven by } d\text{-dimensional Wiener processes}
\]
4 Euler method

\[Y(t) = \int_{t_0}^{t} f_s(s, Y_s) ds + \int_{t_0}^{t} \dot{Y}_s ds, \quad Y(t_0) = y_0.
\]

Therefore, it follows that
\[
\lim_{h \searrow 0} \frac{Y_{n+1}(t, \omega) - Y_{n+1}(t + h, \omega)}{h} = f_\omega(t, Y(t, \omega)).
\]

Similar to (3.19) we can obtain
\[
Y_{n+1}(t-h, \omega) - Y_{n+1}(t, \omega) = (-1) \int_{t-h}^{t} f_\omega(s, Y_n(s, \omega)) ds,
\]

which similar to (3.20) we get
\[
\lim_{h \searrow 0} \frac{Y_{n+1}(t-h, \omega) - Y_{n+1}(t, \omega)}{h} = f_\omega(t, Y(t, \omega)).
\]

Finally, it follows that \(Y_{n+1}\) is (ii)-differentiable with \(P1\) and
\[
Y'_{n+1}(t, \omega) = f(t, Y_n(t, \omega)), \forall t \in [a, b].
\]

Similar to (3.14), we can show that for the sequence \(Y(t, \omega)\) the cauchy convergence condition is satisfied uniformly on the variable \(t\) with \(P1\), and therefore a consequence \(Y_n(t, \omega)\) is uniformly convergent to \(Y(t, \omega)\) for every \(t \in [a, b]\). Obiously, \(Y(t, \cdot)\) is a continuous fuzzy stochastic process. We shall show that \(Y(t, \cdot)\) is a solution of the stochastic fuzzy initial value problem (1.1), since \(f_\omega\) is continuous, using (3.22) for every \(t \in [a, b]\), with \(P1\), as \(n \to \infty\) we have
\[
Y'(t, \omega) = f_\omega(t, Y(t, \omega)).
\]

Therefore, \(Y(t, \omega)\) is a solution of the stochastic fuzzy initial value problem (1.1) for every \(t \in [a, b]\), with \(P1\).

Let us denote the \(\alpha\)-cuts of \(Y(t, \omega)\) by the level sets
\[
[Y(t, \omega)]^\alpha = [\bar{Y}(t, \omega; \alpha), \underline{Y}(t, \omega; \alpha)], \alpha \in [0, 1]
\]
then, we can rewrite the \(\alpha\)-cuts of the relation (3.3) as the form
\[
\begin{cases}
Y_{n+1}(t, \omega; \alpha) = \bar{Y}_0(\omega; \alpha) - (-1) \int_{t_0}^{t} f_\omega(s, Y_n(s, \omega; \alpha)) ds, \\
\underline{Y}_{n+1}(t, \omega; \alpha) = \underline{Y}_0(\omega; \alpha) + (-1) \int_{t_0}^{t} f_\omega(s, \underline{Y}_n(s, \omega; \alpha)) ds,
\end{cases}
\]

with initial conditions \([\bar{Y}(0, \omega; \alpha)] = \bar{Y}_0(\omega; \alpha), [\underline{Y}(0, \omega; \alpha)] = \underline{Y}_0(\omega; \alpha)\).

## 4 Euler method

In this paper we consider numerical methods for the strong solution of fuzzy stochastic differential equations (FSDEs) driven by d-dimensional Wiener processes
\[
dy(t) = f(t, y(t)) dt \oplus \sum_{i=1}^{d} g_j(t, y(t)) \circ dW_j(t), y(t_0) = y_0, t \in [t_0, T],
\]

International Scientific Publications and Consulting Services
where \( f(t, y(t)) \) is the drift coefficient, \( g_j(t, y(t)) \) is the diffusion coefficient and \( W_j(t) \) is the standard Wiener process whose increment \( \Delta W_j(t) = W_j(t + \Delta t) - W_j(t) \) is a Gaussian random variable \( N(0, \Delta t) \) when \( \Delta = 1 \), the FSDEs driven by one Wiener process are given by

\[
dy(t) = f(t, y(t))dt \oplus g(t, y(t)) \odot dW(t), y(t_0) = y_0, \quad t \in [t_0, T],
\]

for solving FODE

\[
y' = f(t, y(t)), \quad y(t_0) = y_0 \in \mathbb{E}, \quad t \in [t_0, T],
\]

if \( f \) to be (i)-differentiable then

\[
\begin{align*}
y(t, r) &= f(t, y(t), r), \quad 0 < r \leq 1 \\
y(t_0, r) &= y_0(r),
\end{align*}
\]

and the explicit Euler method is stable if \( |1 + \lambda h| < 1 \), or if \( \lambda h \in B(-1, 1) \) the circle of radius centred on \((-1, 0)\). Applying the implicit Euler method to the linear test equation, gives

\[
\begin{align*}
y_{n+1} &= (1 + \lambda h)y_n, \quad 0 < r \leq 1 \\
y_{n+1} &= (1 + \lambda h)y_n,
\end{align*}
\]

and the explicit Euler method is stable if \( |1 + \lambda h| < 1 \), or if \( \lambda h \in B(-1, 1) \) the circle of radius centred on \((-1, 0)\). Applying the implicit Euler method to the linear test equation, gives
if \( y \) is (ii) -differentiable then we have
\[
\begin{align*}
\gamma_{n+1} &= \frac{1}{1-\lambda_n} \gamma_n, \\
\delta_{n+1} &= \frac{1}{l} \delta_n,
\end{align*}
\] (4.36)

and the implicit Euler method is stable if \( |1 - \lambda_n| < 1 \), that is stable for any \( h > 0 \)

Consider now what happens in the Fuzzy stochastic case. Applying the implicit Euler method to the fuzzy linear test equation (interpreted in the Itô sense).

\[
dy = ay \odot dt \odot by \odot dW(t), y_0 = y(t_0) \in \mathbb{E}
\]
gives
If \( y \) is (i) -differentiable then we have
\[
\begin{align*}
\gamma_{n+1} &= R(h, \triangle W_n) \gamma_n, \\
\delta_{n+1} &= R(h, \triangle W_n) \delta_n,
\end{align*}
\] (4.37)

if \( y \) is (ii) -differentiable then we have
\[
\begin{align*}
\gamma_{n+1} &= R(h, \triangle W_n) \gamma_n, 0 < r \leq 1 \\
\delta_{n+1} &= R(h, \triangle W_n) \delta_n,
\end{align*}
\] (4.38)

where
\[
R(h, \triangle W_n) = \frac{1}{1 - ah - b \triangle W_n}.
\]

The numerical solution of the implicit Euler method converges to the exact solution of the corresponding right-point FSDE but not that of the Itô SDE.

5 The composite Euler method

Thus in this paper, we introduce a modified method to improve upon the stability properties of the Euler methods. we call this method the composite Euler method and it is a combination of the semi-implicit Euler method and the implicit Euler method, given by
\[
y_{n+1} = y_n \oplus f((t_{n+1}, y_{n+1})h \oplus [\lambda_n g(t_n, y_n) \oplus (1 - \lambda_n) g(t_{n+1}, y_{n+1})] \odot \triangle W_n),
\]
that is,
\[
y_{n+1} = \frac{1 + \lambda_n q I_n}{1 - p - (1 - \lambda_n) q I_n} \odot y_n,
\] (5.39)

where \( p = ah, q = b\sqrt{h}, \triangle W_n = \sqrt{h} I_n \) and \( I_n \) is the nth realization of \( I \), the standard normal random variable \( N(0, 1) \). where \( \lambda_n \in [0, 1] \) and is determined in every step.it becomes the semi-implicit Euler method when \( \lambda_n \equiv 1 \). or the implicit Euler method when \( \lambda_n \equiv 0 \). we can obtain different composite Euler methods if different criteria for choosing \( \lambda_n \) are used. the general principle for selecting \( \lambda_n \) is to ensure
\[
\lim_{N \to \infty} \prod_{n=0}^{N-1} \frac{1 + \lambda_n q I_n}{1 - p - (1 - \lambda_n) q I_n}
\]

converges to 0 as fast as possible when the underlying problem is itself asymptotically stable.Since the solution of (4.36) is (interpreted in the Itô sense) is
\[
y(t) = e^{(a-(1/2)b^2)t+bW(t)}y_0.
\]
The problem is asymptotically stable whenever \( Re(a - \frac{1}{2} b^2) < 0 \). Criterion 1. For solving FSDEs, a simple criterion for selecting \( \lambda_n \) in the composite Euler method is given by

\[
\lambda_n^{(1)} = \begin{cases} 
0, & I_n < 0 \\
1, & I_n \geq 0 
\end{cases} \tag{5.40}
\]

The composite Euler method with Criterion 1 is called the composite Euler method of type 1.

6 Convergence properties

The solution of FSDE (2.2) can be written as

\[
y(t) = y_0 \oplus \int_0^t f(s, y(s)) ds \oplus \int_0^t g(s, y(s)) \circ dW(s)
\]

where the integral \( \int_0^t g(s, y(s)) \circ dW(s) \) is a fuzzy stochastic integral. This integral can be calculated by the limit of the approximating sums. Giving an equidistant discretization of interval \( [0, T] \).

\[
t_0 < t_1 < \ldots < t_N = T, t_n = t_0 + \frac{(T - t_0)n}{N}, n = 0, 1, 2, \ldots, N. \tag{6.41}
\]

and letting \( \xi_n = \theta_n + (1 - \theta) t_{n-1} (\theta \in [0, 1]) \), the fuzzy stochastic integral is defined as the limit (in the mean-square sense), as \( N \to \infty \), of the approximating sums

\[
\sum_{n=1}^N g(\xi_n) \circ (W(t_n) - W(t_{n-1}))
\]

Unlike Riemann integrals, the value of fuzzy stochastic integrals depend on the choice of \( \theta \). For example, the fuzzy stochastic integral of the Wiener process is given by

\[
\int_{t_0}^T W(s) dW(s) = \frac{1}{2} (W^2(T) - W^2(t_0)) + (\theta - \frac{1}{2})(T - t_0).
\]

We have the following three types of fuzzy stochastic integrals and corresponding FSDEs: The Itô integrals when \( \theta = 0 \). The corresponding Itô FSDEs are the equations using the usual notation (2.2); The Stratonovich integrals when \( \theta = \frac{1}{2} \). The corresponding Stratonovich FSDEs are denoted by

\[
dy(t) = f_1(y(t)) dt \oplus g(y(t)) \circ dW(t)
\]

The backward integrals when \( \theta = 1 \). The corresponding right-point FSDEs[...] are denoted by

\[
dy(t) = f_2(y(t)) dt \oplus g(y(t)) \circ dW(t)
\]

The relationships between these FSDEs are

\[
f_1(y(t)) = f(y(t)) \circ \frac{1}{2} g'(y(t)) \circ g(y(t)),
\]

\[
f_2(y(t)) = f(y(t)) \circ g'(y(t)) \circ g(y(t)),
\]

It can be proved that the numerical solutions of the implicit Euler method for solving right-point FSDEs converge to the exact solutions of the right-point FSDEs with order \( O(h^2) \). Now, we consider the convergence properties of the composite Euler method applying to the linear test equation ( ), given by

\[
y_N = \frac{1 + \lambda_{N-1} b_{N-1}}{1 - ah - (1 - \lambda_{N-1} b_{N-1})} \circ y_{N-1} = y_0 \oplus \prod_{n=1}^{N-1} \frac{1 + \lambda_n b_n}{1 - ah - (1 - \lambda_n) b_n}.
\]

Let
\[ P_N = \prod_{n=0}^{N-1} \frac{1 + \lambda_N b_n}{1 - ah - (1 - \lambda_n)b_n}, \]

then

\[ \ln P_N = \sum_{n=0}^{N-1} \ln(1 + \lambda_n b_n) - \sum_{n=0}^{N-1} \ln(1 - ah - (1 - \lambda_n)b) \triangle W_n, \]

finally, we have that

\[ \ln P_N = a(t_N \ominus t_0)(W(t_N) - W(t_0)) \oplus \frac{1}{2} \sum_{n=0}^{N-1} (1 - 2\lambda_n)(b \triangle W_n)^2 \oplus R_N. \]

It can be proved that \( R_N \) converges to zero in mean-square sense, namely

\[ \lim_{N \to \infty} E(R_N^2) = 0 \]

This means that the numerical solution of composite Euler method of type 1 converge to the exact solution of the Stratonovich linear test equation, if \( f \) (i)-differential then

\[ \lim_{N \to \infty} E[(y_N \ominus y_0 \ominus e^{a(T-t_0) + h(W(T)-W(t_0))}^2)] = 0 \]

and

if \( f \) (ii)-differential then

\[ \lim_{N \to \infty} E[(y_N \ominus y_0 \ominus e^{-a(T-t_0) + h(W(T)-W(t_0))}^2)] = 0 \]

7 Examples

In this section, we consider several examplese given solutions of the Euler method for fuzzy Stiff stochastic differential equation under generalized differential ability.

Example 7.1. Let us consider the first equation is the Fuzzy linear test equation \( a = -1, b = 1 \). Fuzzy Itô form:

\[
\begin{align*}
\begin{cases}
\frac{dy}{dt} = -dt + dW(t) & (0 \leq t \leq 3) \\
y(0) = \hat{1}.
\end{cases}
\end{align*}
\]

ify is (i)-differential then exact solution

\[ y(t, r) = \left( \begin{array}{c} y_0(r) e^{3t + W(t)} \\
y_0(r) e^{3t + W(t)} \end{array} \right). \]

If \( y \) is (ii)-differential then exact solution is

\[ y(t, r) = \left( \begin{array}{c} y_0(r) e^{-3t + W(t)} \\
y_0(r) e^{-3t + W(t)} \end{array} \right). \]

where \( y \) is (i)-differential then length support solution \( y(t) \) \( \text{wh} \ t \to \infty \text{ then } y(t) \to +\infty \). Therefore,where \( y \) is (ii)-differential then length support solution \( y(t) \) \( \text{when} \ t \to \infty \text{ then } y(t) \to 0 \).

Example 7.2. Let us consider the first equation is the Fuzzy linear test equation \( a = -1, b = 1 \). Fuzzy Stratonovich’s form:

\[
\begin{align*}
\begin{cases}
\frac{dy}{dt} = -dt + d \circ W(t) & (0 \leq t \leq 3) \\
y(0) = \hat{1} = (r, -2 - r) & (0 < t \leq 1).
\end{cases}
\end{align*}
\]
If $y$ is (i)-differential then exact solution

$$y(t, r) = \left( y_0(r)e^{t+W(t)}; y_0(r)e^{t+W(t)} \right).$$

If $y$ is (ii)-differential then the exact solution is

$$y(t, r) = \left( y_0(r)e^{-t+W(t)}; y_0(r)e^{-t+W(t)} \right).$$

where $y$ is (i)-differential then length support solution $y(t)$ when $t \to \infty$ then $y(t) \to \infty$. Therefore, where $y$ is (ii)-differential then length support solution $y(t)$ when $t \to \infty$ then $y(t) \to 0$.

8 Conclusion

Developing convergence and stability properties Euler method, is provided solutions to fuzzy stochastic initial-value problems for first order linear fuzzy stochastic differential equations which is interpreted by using generalized differentiability concept. This may confer solutions which have a decreasing length of their support.

References

   http://dx.doi.org/10.1016/j.fss.2004.08.001

   http://dx.doi.org/10.1016/j.ins.2006.08.021

   http://dx.doi.org/10.7151/dmdico.1125

   http://dx.doi.org/10.1016/j.fss.2009.02.003

   http://dx.doi.org/10.1016/j.na.2010.04.049

   http://dx.doi.org/10.1016/j.nonrwa.2011.08.022

   http://dx.doi.org/10.1016/S0165-0114(98)00355-8

   http://dx.doi.org/10.1016/0022-247X(86)90093-4


