Characterization of Fuzzy Graphs into different categories using Arcs in Fuzzy Graphs

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Abstract
The relationships between different types of arcs in both regular and totally regular fuzzy graphs are analyzed in this paper. In these categories, the crisp graph obtained is nothing but a cycle. A characterization of totally regular fuzzy graph is obtained using arcs.

Keywords: $\alpha, \beta, \delta$ - arcs, fuzzy cycle, regular fuzzy graph, totally regular fuzzy graph, weak totally regular fuzzy graph, strong totally regular fuzzy graph.

1 Introduction
Fuzzy graphs were introduced by Rosenfeld [5], ten years after Zadeh’s seminal paper “Fuzzy Sets”[6]. Fuzzy graph theory is finding numerous application in the fields of information theory, neural networks, expert systems, cluster analysis, medical diagnosis, control theory, etc. Fuzzy modeling has become an efficient tool. Fuzzy models give more precision, flexibility and compatibility to the system when compared to classical models [6, 7]. Rosenfeld obtained the fuzzy analogues of several basic graph theoretic concepts like bridges, paths, cycles, trees and connectedness and established some of their properties [5]. Bhattacharya [8] had established some connectivity concepts regarding fuzzy cut nodes and fuzzy bridges. He has also introduced fuzzy groups and metric notion in fuzzy graphs. Bhutani [4] had studied automorphisms on fuzzy graphs and certain properties of complete fuzzy graphs. Bhattacharya and Suraweera had introduced an algorithm to find the connectivity of a pair of nodes in a fuzzy graph [3]. Nagoor Gani and Radha [1] defined regular and totally regular fuzzy graph. They stated some necessary and sufficient condition for these type of graphs. Sunil Mathew and Sunitha defined different types of arcs in fuzzy graphs and using them classified fuzzy graphs [2]. In this paper, a comparative study is made between regular and totally regular fuzzy graphs with reference to different types of arcs in fuzzy graphs. Also a necessary condition for a graph to be regular or totally regular is formulated in terms of arcs.

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2 Preliminaries and notations

Definition 2.1. Fuzzy Graph: A fuzzy graph G is a pair of functions G:(σ,µ) where σ is a fuzzy subset of a non empty set V and µ is a symmetric fuzzy relation on σ. The underlying crisp graph of G:(σ,µ) is denoted by G*(V,E) where E ⊆ V × V [5].

Definition 2.2. Complete Fuzzy Graph: A Fuzzy Graph G is complete if µ(uv) = σ(u) ∧ σ(v) for all u,v ∈ V, where uv denotes the edge between u and v [8].

Definition 2.3. Vertex Degree: Let G:(σ,µ) be a fuzzy graph. The degree of a vertex u is d_G(u) = ∑ µ(uv) [5].

Definition 2.4. Fuzzy Cycle: Let G:(σ,µ) be a fuzzy graph such that G*(V,E) is a cycle. Then G is a fuzzy cycle if and only if there does not exist a unique edge xy such that µ(xy)=∪{µ(uv)/uv} >0 [5].

Definition 2.5. Regular Fuzzy Graph: Let G:(σ,µ) be a fuzzy graph on G*(V,E). If d_G(v) = k for all v ∈ V, (i.e) if each vertex has same degree k, then G is said to be a regular fuzzy graph of degree k or a k-regular fuzzy graph [1].

Definition 2.6. Totally Regular Fuzzy Graph: Let G:(σ,µ) be a fuzzy graph on G*(V,E). The total degree of a vertex u ∈ V is defined by td_G(u) = ∑ µ(uv) + σ(u) = d_G(u) + σ(u)

If each vertex of G has the same degree k, then G is said to be a totally regular fuzzy graph of total degree k or k-totally regular fuzzy graph [1].

Definition 2.7. Strength of a path: If u,v are nodes in G and if they are connected by means of a path then the strength of that path is defined as

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i=1 µ(v_i-1, v_i) i.e, it is the strength of the weakest arc [8].

Definition 2.8. µ^k(u,v): If u, v are connected by means of paths of length k then µ^k(u,v) is defined as

µ^k(u,v) = sup{µ(u,v_i) ∧ µ(v_{i+1}, v_{i+2}) ∧ ... ∧ µ(v_{k-1}, v_k) / u,v_i,v_{i+2} ..., v_k ∈ V} [8].

Definition 2.9. µ^(uv): The Strength of connectedness between u and v is defined as

µ^(uv) = sup{µ^k(u,v) / k = 1,2,3,...} and is denoted as conn(u,v) [8].

Definition 2.10. α - arc: An arc (xy) in G is called an α - arc if µ(xy) ≥ conn_G−(xy)(xy) [2].

Definition 2.11. β - arc: An arc (xy) in G is called an β - arc if µ(xy) = conn_G−(xy)(xy) [2].
Definition 2.12. **δ – arc**: An arc \( (xy) \) in \( G \) is called a \( \delta \)-arc if \( \mu(xy) < \text{conn}_{G-}(xy) \) [2].

**Theorem 2.1.** “A Fuzzy Graph \( G \) whose crisp graph is an odd cycle is regular if and only if \( \mu \) is a constant function” [1].

**Theorem 2.2.** “A fuzzy graph \( G \) whose crisp graph is an even cycle is regular if and only if \( \mu \) is a constant function or alternate edges will have same values” [1].

3 Main Section

**Theorem 3.1.** A regular fuzzy graph \( G : (\sigma, \mu) \) whose odd cycle is the crisp graph \( G^* : (V, E) \) contains only \( \beta \)-arc.

**Proof.** If \( G \) contains only \( \beta \)-arc then by the definition (2.11), we have, \( \mu(x, y) = \text{conn}_{G-}(x, y)(x, y) \)

Thus all the edges in \( G \) will have the same membership value.

Now from Theorem 2.1, we get \( G \) as a regular fuzzy graph.

Conversely, let \( G \) be a regular fuzzy graph then by Theorem 2.1, \( \mu \) is a constant function. Thus the deletions of any arc in \( G \) will not affect the strength of connectivity of any u-v paths in \( G \).

i.e., \( \mu(x, y) = \text{conn}_{G-}(x, y)(x, y), \forall(x, y) \in G \)

Thus \( G \) contains only \( \beta \)-arc.

**Remark 3.1.** The above theorem does not hold for totally regular fuzzy graph.

For example, consider \( G : (\sigma, \mu) \) where \( \sigma(x) = 0, \; \sigma(y) = 0.6, \; \sigma(z) = 0.8, \; \mu(xy) = 0.5, \; \mu(yz) = 0.4 \) and \( \mu(zx) = 0.3 \)

![Figure 1: Totally regular fuzzy graph with \( \alpha \) and \( \delta \) arcs](image)

Here \( G \) is totally regular fuzzy graph. But it contains \( \alpha \) and \( \delta \) arcs.

**Theorem 3.2.** A regular fuzzy graph \( G : (\sigma, \mu) \) whose even cycle is the crisp graph \( G^* : (V, E) \) contains \( \alpha \) and \( \beta \)-arcs, but no \( \delta \)-arc.
**Proof.** Assume $G$ has no $\delta$-arc, then by definition (2.10) and definition (2.11) we have, $\mu(x, y) \geq \text{conn}_{G} (x, y)$ which implies that $\mu$ has either constant values or alternative edges will have same values. Now from Theorem 2.2 we get $G$ as a regular fuzzy graph.

Conversely, Let $G$ be a regular fuzzy graph then by Theorem 2.2. The values of $\mu$ is either constant or alternate edges will have same values. i.e., $\mu(x, y) \geq \text{conn}_{G} (x, y)$.

Thus $G$ contains either $\alpha$ or $\beta$-arcs, but no $\delta$-arc.

**Remark 3.2.** The above theorem does not hold for totally regular fuzzy graph. For example, consider $G: (\sigma, \mu)$ where $\sigma(p) = 0.8$, $\sigma(q) = 0.6$, $\sigma(r) = 0.4$, $\sigma(s) = 0.6$, $\mu(pq) = 0.2$, $\mu(qr) = 0.3$, $\mu(rs) = 0.4$ and $\mu(sp) = 0.1$.

![Figure 2: Totally regular fuzzy graph with $\delta$ arc](image)

Here $G$ is totally regular but it contains $\delta$ arc.

**Definition 3.1.** A totally regular fuzzy graph with $\delta$-arc in its $u$-$v$ path is said to be weak totally regular fuzzy graph.

**Remark 3.3.** Consider $G: (\sigma, \mu)$ where $\sigma(p) = \sigma(q) = \sigma(r) = \sigma(s) = 0.3$, $\mu(pq) = 0.2$, $\mu(qr) = 0.3$, $\mu(rs) = 0.2$ and $\mu(sp) = 0.3$.

![Figure 3: Totally regular fuzzy graph without $\delta$ arc](image)
Also, consider \( G : (\sigma, \mu) \) where \( \sigma(p) = \sigma(q) = \sigma(r) = \sigma(s) = 0.4, \mu(pq) = 0.6, \mu(qr) = 0.6, \mu(rs) = 0.6 \) and \( \mu(sp) = 0.6 \).

![Figure 4: Totally regular fuzzy graph without \( \delta \) arc]

**Definition 3.2.** A totally regular fuzzy graph without \( \delta \)-arc in its \( u-v \) path is said to be strong totally regular fuzzy graph.

**Theorem 3.3.** A regular fuzzy graph \( G : (\sigma, \mu) \) with its crisp graph \( G^* : (V, E) \) as even cycle is both regular and totally regular if it does not contain \( \delta \)-arc.

**Proof.**

**Case (i)** Let \( G \) be both regular and totally regular fuzzy graph with constant values in \( \sigma \) and \( \mu \) then by the definition (2.11) \( G \) contains only \( \beta \)-arc.

**Case (ii)** Let \( G \) be both regular and totally regular fuzzy graph with constant values in \( \sigma \) and with same alternate values in \( \mu \) then by the definition (2.10) and definition (2.11), \( G \) contains only \( \alpha \) and \( \beta \)-arcs but no \( \delta \)-arc.

**4 Conclusion**

We have noted down the existence of different types of arcs in fuzzy graphs. We used the characteristics of different types of arcs to categorize Regular and Totally Regular Fuzzy Graphs. We have also observed that the presence of different types of arcs make the graph either weak or strong totally Regular Fuzzy Graph. Further analysis, may lead us to a better understanding of the nature of Fuzzy Graphs.

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References


   http://dx.doi.org/10.1016/j.ins.2009.01.003

   http://dx.doi.org/10.1016/0167-8655(91)90307-8

   http://dx.doi.org/10.1016/0167-8655(89)90049-4


   http://dx.doi.org/10.1016/j.ins.2005.01.017

   http://dx.doi.org/10.1016/j.ins.2008.02.012

   http://dx.doi.org/10.1016/0167-8655(87)90012-2