Uzawa method for fuzzy linear system

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Abstract
An Uzawa method is presented for solving fuzzy linear systems whose coefficient matrix is crisp and the right-hand side column is arbitrary fuzzy number vector. The explicit iterative scheme is given. The convergence is analyzed with convergence theorems and the optimal parameter is obtained. Numerical examples are given to illustrate the procedure and show the effectiveness and efficiency of the method.

Keywords: Uzawa method; Fuzzy linear system (FLS).

1 Introduction
A fuzzy linear system (FLS) is a linear system whose parameters are all or partially represented by fuzzy numbers. FLSs have many applications in control problems, information, physics, statistics, engineering, economics, finance and even social sciences. Therefore, it’s important to establish mathematical models and numerical methods for fuzzy linear systems. In the 1990s, Buckley et al. [13, 14, 15] investigated fuzzy equations in series. Hereafter, more and more people devote to studying fuzzy linear systems. Rao and Chen [24] considered the numerical solutions of FLSs in engineering analysis. Allahviranloo et al. proposed the fuzzy symmetric solutions and other algebraic solution for fuzzy linear systems [6, 11], and the solutions of fully fuzzy linear systems [8, 9, 10, 12, 18, 19, 23]. Friedman et al. [21] suggested a general model for solving a class of $n \times n$ FLSs

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= y_1, \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= y_2, \\
    \vdots \\
    a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= y_n,
\end{align*}
\]

where the coefficient matrix $A = (a_{ij})$ is a crisp matrix and $y_i$ is a fuzzy number, $1 \leq i, j \leq n$. Many authors study numerical iterative methods for solving FLS (1.1), such as Abbasbandy [1, 2], Allahviranloo [3, 4, 5], Dehghan and Hashemi [17], Fariborz Araghi and Fallahzadeh [20], Miao, Wang and Zheng [22, 25, 26]. In this paper, an Uzawa method (cf. [16]) is provided for solving FLS (1.1) numerically. The paper is organized as follows. In Section 2, some basic definitions and results about fuzzy number and FLS are recalled. In Section 3, we propose the Uzawa method with the convergence theorems. The illustrated numerical examples are given in Section 4 and the conclusion is in Section 5.

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2 Preliminaries

Following [21], a fuzzy number is defined as \((u(r), \Pi(r))\), \(0 \leq r \leq 1\), which satisfies,

- \(u(r)\) is a bounded left continuous nondecreasing function over \([0, 1]\),
- \(\Pi(r)\) is a bounded left continuous nonincreasing function over \([0, 1]\),
- \(u(r) \leq \Pi(r), 0 < r < 1\).

To define a solution to the system (1.1) we should recall the arithmetic operations of arbitrary fuzzy numbers
\(x = (\underline{x}(r), \overline{x}(r)), y = (\underline{y}(r), \overline{y}(r)), 0 \leq r < 1\), and real number \(k\),

1. \(x = y\) if and only if \(\underline{x}(r) = \underline{y}(r)\) and \(\overline{x}(r) = \overline{y}(r)\),
2. \(x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))\), and
3. \(kx = \begin{cases} (k\underline{x}(r), k\overline{x}(r)), & k \geq 0, \\ (k\overline{x}(r), k\underline{x}(r)), & k < 0. \end{cases}\)

**Definition 2.1.** A fuzzy number vector \(X = (x_1, x_2, \ldots, x_n)^T\) given by
\(x_i = (\underline{x}(r), \overline{x}(r)), 1 \leq i \leq n, 0 \leq r < 1\),
is called a solution of the fuzzy linear system (1.1) if
\[
\begin{cases}
\sum_{j=1}^{n} a_{ij} x_j = \sum_{j=1}^{n} a_{ij} x_j = y_i, \\
\sum_{j=1}^{n} a_{ij} x_j = \sum_{j=1}^{n} a_{ij} x_j = \overline{y}_i.
\end{cases}
\] (2.2)

With (2.2), Friedman et al. [21] extend FLS (1.1) to a \(2n \times 2n\) crisp linear system
\(SX = Y\) (2.3)
where \(S = (s_{ij}), s_{kl}\) are determined as follows
\[a_{ij} \geq 0 \Rightarrow s_{ij} = a_{ij}, \quad a_{ij} < 0 \Rightarrow s_{ij} = -a_{ij};\]
and any \(s_{ij}\) which is not determined by the above items is zero, \(1 \leq k, l \leq 2n,\) and

\[X = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\overline{x}_1 \\ \vdots \\ -\overline{x}_n \end{bmatrix}, \quad Y = \begin{bmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_n \\ \overline{y}_1 \\ \vdots \\ \overline{y}_n \end{bmatrix}.
\]

What’s more, the matrix \(S\) has the structure
\[
\begin{bmatrix}
S_1 & S_2 \\
S_2 & S_1
\end{bmatrix}
\]
where \(S_1, S_2 \geq 0, A = S_1 - S_2\), and (2.3) can be rewritten as follows
\[
\begin{cases}
S_1 X - S_2 X = Y, \\
S_2 X - S_1 X = \overline{Y},
\end{cases}
\] (2.4)
where
\[
X = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ \overline{x}_1 \\ \vdots \\ \overline{x}_n \end{bmatrix}, \quad Y = \begin{bmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_n \\ \overline{y}_1 \\ \vdots \\ \overline{y}_n \end{bmatrix}, \quad \overline{Y} = \begin{bmatrix} -\overline{y}_1 \\ \vdots \\ -\overline{y}_n \end{bmatrix}.
\]

The following theorem indicates when FLS (1.1) has a unique solution.
Theorem 2.1. [21]. The matrix $S$ is nonsingular if and only if the matrices $A = S_1 - S_2$ and $S_1 + S_2$ are both nonsingular.

Under the conditions of theorem (2.1), the solution $X = S^{-1}Y$ of (1.1) is thus unique but may still not be an appropriate fuzzy vector. By Theorem 2 of [21], we know that $S^{-1}$ has the same structure like $S$, i.e. $S^{-1} = \begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix}$.

The following result provides a sufficient condition for the unique solution to be a fuzzy vector.

Theorem 2.2. [21]. The unique solution $X$ of (2.3) is a fuzzy vector for arbitrary fuzzy vector $Y$, if $S^{-1}$ is nonsingular.

Restricting the discussion to triangular fuzzy numbers, i.e. $\mu_i(r), \nu_i(r)$ and consequently $\alpha_i(r), \beta_i(r)$ are all linear functions of $r$, and having calculated $X$ which solves (2.3), most of the authors use the definition of the fuzzy solution to the original system given by (1.1) as follows.

Definition 2.2. Let $X = \{(\alpha_i(r), \beta_i(r)), 1 \leq i \leq n\}$ denote the unique solution of (2.3). The fuzzy number vector $U = \{(\mu_i(r), \nu_i(r)), 1 \leq i \leq n\}$ defined

\[
\mu_i(r) = \min\{\alpha_i(r), \beta_i(r), \alpha_1(1), \beta_1(1)\}, \\
\nu_i(r) = \max\{\alpha_i(r), \beta_i(r), \alpha_1(1), \beta_1(1)\}
\]

is called the fuzzy solution of $SX = Y$. If $(\alpha_i(r), \beta_i(r)), 1 \leq i \leq n$ are all fuzzy numbers then $u_i(r) = \alpha_i(r), \nu_i(r) = \beta_i(r), 1 \leq i \leq n$ and $U$ is called a strong fuzzy solution; otherwise, $U$ is called a weak fuzzy solution. Recently, however, Allahviranloo et al. [7] showed that this definition is not always true, that is, it doesn’t always produce a fuzzy number vector. Thus, this research line needs further investigation.

3 The Uzawa method for FLS

For the case $S$ is nonsingular, without loss of generality, assume that $s_{ii} > 0, i = 1, 2, \ldots, 2n$, by (2.4), we can get the Uzawa iterative scheme as follows,

\[
X_{k+1} = X_k + S_1^{-1}(S_2X_k - S_1X_k + Y), \\
\frac{X_{k+1}}{X_k} = \frac{X_k}{X_k} + \tau(S_2X_k - S_1X_k - Y),
\]

where $\tau$ is a real parameter, and in matrix form,

\[
\begin{bmatrix}
X_{k+1} \\
X_k
\end{bmatrix} = H_\tau \begin{bmatrix}
X_k \\
X_k
\end{bmatrix} + B_\tau \begin{bmatrix}
Y \\
-Y
\end{bmatrix},
\]

in which $H_\tau = \begin{bmatrix} 0 & S_{-1}S_2 \\ 0 & I - \tau(S_1 - S_2S_{-1}S_2) \end{bmatrix}$, $B_\tau = \begin{bmatrix} S_{-1} & 0 \\ \tau S_2S_{-1} & \tau I \end{bmatrix}$. Analyzing (3.6), we can get the following convergence theorem:

Theorem 3.1. Let $\lambda$ denote an arbitrary eigenvalue of $S_1 - S_2S_{-1}S_2$ and $\text{Re}\lambda$ its real part. The Uzawa method (3.5) converges if

1. $\max\{\text{Re}\lambda\} > \min\{\text{Re}\lambda\} \geq 0$ and $0 < \tau < \min\{2\text{Re}\lambda / |\lambda|^2\}$;
2. $\min\{\text{Re}\lambda\} < \max\{\text{Re}\lambda\} \leq 0$ and $\max\{2\text{Re}\lambda / |\lambda|^2\} < \tau < 0$.

Proof. Let $\mu$ be an arbitrary eigenvalue of $H_\tau$. Then we have $\mu = 0$ or $\mu = 1 - \tau\lambda$. So the iterative scheme (3.5) converges if $|1 - \tau\lambda| < 1$, with $\text{Im}\lambda$ denoting the imaginary part of $\lambda$, that is

\[
(1 - \tau\text{Re}\lambda)^2 + (\tau\text{Im}\lambda)^2 < 1,
\]

i.e.,

\[
\tau(|\lambda|^2 \tau - 2\text{Re}\lambda) < 0,
\]
4 Numerical Examples

Converge slowly.

\[ \begin{align*}
0 < \tau &< 2 \text{Re} \lambda / |\lambda|^2, \quad \text{Re} \lambda > 0, \\
2 \text{Re} \lambda / |\lambda|^2 &< \tau < 0, \quad \text{Re} \lambda < 0.
\end{align*} \]

Then we have the Uzawa method (3.5) converges if

1. \( \lambda_M > \lambda_m \geq 0 \) and \( 0 < \tau < 2 / \lambda_M \);
2. \( \lambda_m < \lambda_M \leq 0 \) and \( 2 / \lambda_m < \tau < 0 \).

\begin{proof}
Proof. Eliminating \( X_{k+1} \) in the second equation in (3.5) we get

\[ X_{k+1} = X_k + \tau [(S_2 S_1^{-1} Y - Y) - (S_1 - S_2 S_1^{-1} S_2) X_k]. \] (3.7)

Obviously, (3.7) is the Richardson iteration for the Schur complement system of (1.1):

\[ (S_1 - S_2 S_1^{-1} S_2) X = S_2 S_1^{-1} Y - Y. \] (3.8)

Let \( e^X_k = \overline{X} - X \), be the iteration error, then from (3.7) and (3.8) we have

\[ e^X_{k+1} = \tau [I - \tau (S_1 - S_2 S_1^{-1} S_2)] e^X_k. \]

As the spectral radius \( \rho \) of \( I - \tau (S_1 - S_2 S_1^{-1} S_2) \) is \( \rho = \max \{|1 - \lambda_m \tau|, |\lambda_M \tau - 1|\} \), the optimal \( \tau \) is \( \tau_{\text{opt}} = \frac{2}{\lambda_M + \lambda_M} \).
\end{proof}

\begin{remark}
The optimal spectral radius of \( I - \tau (S_1 - S_2 S_1^{-1} S_2) \) is \( \rho_{\text{opt}} = \frac{\kappa - 1}{\kappa + 1} \), where \( \kappa = \lambda_M / \lambda_m \) is the condition number of \( S_1 - S_2 S_1^{-1} S_2 \). If this operator is not well conditioned, i.e., \( \kappa \) is large, then the above iterative method may converge slowly.
\end{remark}

4 Numerical Examples

\begin{example}
[25]. Consider 2 \times 2 fuzzy linear system

\[ \begin{align*}
&x_1 - x_2 = (r/2 - r), \\
x_1 + 3x_2 = (4 + r, 7 - 2r).
\end{align*} \]

The exact solution is

\[ \begin{align*}
x_1 &= (x_1(r), x_1(r)) = (1.375 + 0.625r, 2.875 - 0.875r), \\
x_2 &= (x_2(r), x_2(r)) = (0.875 + 0.125r, 1.375 - 0.375r).
\end{align*} \]

Direct calculations yield \( \lambda_m = 0.8453, \lambda_M = 3.1547, \tau_{\text{opt}} = 0.5, \kappa = 3.7321 \).

With \( \tau = \tau_{\text{opt}} = 0.5 \) and \( X_0 = [0, 0, 0, 0]^{T} \), after 17 iterations, the numerical solution is

\[ X_{17} = \begin{bmatrix}
1.3748 + 0.6251r \\
0.8751 + 0.1250r \\
2.8748 - 0.8749r \\
1.3753 - 0.3751r
\end{bmatrix}, \]
\end{example}

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i.e.,
\[
\begin{align*}
&\begin{cases}
x_1 = (1.3748 + 0.6251r, 2.8748 - 0.8749r), \\
x_2 = (0.8751 + 0.1250r, 1.3753 - 0.3751r),
\end{cases}
\end{align*}
\]
and the Hausdorff distance is 3.3760e-004 between the approximate and exact solutions. If take \(\tau = 0.2\) and \(0.6\), and \(X_0 = 0\), after 17 iterations, the numerical solutions are
\[
\begin{align*}
&\begin{cases}
x_1 = (1.4345 + 0.6067r, 2.7684 - 0.8423r), \\
x_2 = (0.8552 + 0.1311r, 1.4245 - 0.3902r),
\end{cases}
\end{align*}
\]
and
\[
\begin{align*}
&\begin{cases}
x_1 = (0.9629 + 0.7438r, 2.9319 - 0.8914r), \\
x_2 = (1.0124 + 0.0854r, 1.7429 - 0.4811r),
\end{cases}
\end{align*}
\]
and the Hausdorff distances are 0.1115 and 0.4228, respectively. We can see that the approximate solution with \(\tau = \tau_{\text{opt}}\) is nearest to the exact solution. As \(\kappa\) is not very large, the convergence rate is pretty good.

**Example 4.2.** [11]. Consider 5 \(\times\) 5 fuzzy linear system
\[
\begin{align*}
&\begin{cases}
6x_1 + x_2 + 3x_3 - x_4 + 6x_5 = (1 + r, 3 - r), \\
5x_1 + 9x_2 + x_3 + 2x_4 + 3x_5 = (6 + r, 8 - r), \\
2x_1 + 3x_2 + 9x_3 + 2x_4 + 3x_5 = (5 + r, 7 - r), \\
-x_1 + x_2 + 3x_3 + 8x_4 + 3x_5 = (3 + r, 5 - r), \\
x_1 + 2x_2 + 2x_3 + x_4 + 9x_5 = (2 + r, 4 - r).
\end{cases}
\end{align*}
\]
The exact solution is
\[
\begin{align*}
x_1 &= (-0.0409 + 0.0462r, 0.0515 - 0.0462r), \\
x_2 &= (0.6130 + 0.0388r, 0.6907 - 0.0388r), \\
x_3 &= (0.3194 + 0.0465r, 0.4124 - 0.0465r), \\
x_4 &= (0.1854 + 0.0671r, 0.3196 - 0.0671r), \\
x_5 &= (-0.0011 + 0.0769r, 0.1581 - 0.0796r).
\end{align*}
\]
Direct calculations yield \(\lambda_{\text{up}} = 5.3160, \lambda_{\text{inf}} = 16.9188, \tau_{\text{opt}} = 0.0899, \kappa = 3.1826\).
With \(\tau = \tau_{\text{opt}} = 0.0899\) and \(X_0 = 0\), after 13 iterations, the numerical solution is
\[
\begin{align*}
&\begin{cases}
x_1 = (-0.0418 + 0.0461r, 0.0451 - 0.0477r), \\
x_2 = (0.6138 + 0.0390r, 0.6939 - 0.0374r), \\
x_3 = (0.3196 + 0.0465r, 0.4147 - 0.0465r), \\
x_4 = (0.1843 + 0.0668r, 0.3162 - 0.0675r), \\
x_5 = (-0.0010 + 0.0796r, 0.1590 - 0.0796r),
\end{cases}
\end{align*}
\]
and the Hausdorff distance is 0.0073 between the approximate and exact solutions.
If take \(\tau = 0.05\) and \(0.1\) and \(X_0 = 0\), after 13 iterations, the numerical solutions are
\[
\begin{align*}
&\begin{cases}
x_1 = (-0.0418 + 0.0461r, 0.0451 - 0.0477r), \\
x_2 = (0.6138 + 0.0390r, 0.6939 - 0.0374r), \\
x_3 = (0.3196 + 0.0465r, 0.4147 - 0.0465r), \\
x_4 = (0.1843 + 0.0668r, 0.3162 - 0.0675r), \\
x_5 = (-0.0010 + 0.0796r, 0.1590 - 0.0796r),
\end{cases}
\end{align*}
\]
and
\[
\begin{align*}
&\begin{cases}
x_1 = (-0.0418 + 0.0461r, 0.0451 - 0.0477r), \\
x_2 = (0.6138 + 0.0390r, 0.6939 - 0.0374r), \\
x_3 = (0.3196 + 0.0465r, 0.4147 - 0.0465r), \\
x_4 = (0.1843 + 0.0668r, 0.3162 - 0.0675r), \\
x_5 = (-0.0010 + 0.0796r, 0.1590 - 0.0796r),
\end{cases}
\end{align*}
\]
and the Hausdorff distances are 0.0094 and 0.0075, respectively. We can see that the approximate solution with \(\tau = \tau_{\text{opt}}\) is nearest to the exact solution. As \(\kappa\) is not very large, the convergence rate is pretty good.
5 Conclusion

We present an Uzawa iterative method for $n \times n$ fuzzy linear system. If the proposed matrix $S$ by Friedman et al. [21] is nonsingular, then for any initial vector $X_0$, the Uzawa iteration will converge to the unique solution of $SX = Y$. The numerical examples show that the method is effective, and with the optimal parameter $t_{opt}$, it is expected to achieve higher accuracy or faster convergence rate than the known methods which have no parameters.

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