Solution of the first order linear fuzzy differential equations by some reliable methods

Mojtaba Ghanbari

Department of Mathematics, Aliabad Katoul Branch, Islamic Azad University, Aliabad Katoul, Iran.

Abstract

Fuzzy differential equations are used in modeling problems in science and engineering. For instance, it is known that the knowledge of dynamical systems modeled by ordinary differential equations is often incomplete or vague. While, fuzzy differential equations represent a proper way to model dynamical systems under uncertainty and vagueness. In this paper, two methods for solving first order linear fuzzy differential equations under generalized differentiability are proposed and compared. These methods are variational iteration method (VIM) and Adomian decomposition method (ADM). The comparison of the exact solutions with solutions obtained by VIM and ADM are in details. The comparison shows that solutions are excellent agreement.

Keywords: Fuzzy numbers; Fuzzy differential equations; Generalized differentiability; Variational iteration method; Adomian decomposition method

1 Introduction

Fuzzy differential equations play an important role in an increasing number of system models in biology, engineering, physics and other sciences. For example, in population models [29], civil engineering [38], bioinformatics and computational biology [18], quantum optics and gravity [36] and in modeling hydraulic [16]. First order linear fuzzy differential equations are one of the simplest fuzzy differential equations which may appear in many applications. However the form of such an equation is very simple, it raises many problems since under different fuzzy differential equation concepts, the behaviour of the solutions is different [14]. Since the fuzzy derivative is used in a fuzzy differential equation, it is natural to begin by presenting a background of fuzzy derivative.
The concept of the fuzzy derivative was first introduced by Chang and Zadeh [20]. Later, Dubois and Prade [22] presented a concept of the fuzzy derivative based on the extension principle. Other methods have been discussed by Puri and Ralescu [39], Goetschel and Voxman [27], Seikkala [40] and Friedman et al. [25, 33]. Buckley and Feuring [17] compare various derivatives of fuzzy function that have been presented in the various literature by comparing the different solutions one may obtain to the fuzzy differential equations using these derivatives. Recently, Bede introduced a strongly generalized differentiability of fuzzy functions in [13] and studied in [14, 15]. The numerical methods for solving fuzzy differential equations are introduced in [3, 4, 9, 10]. In 2009, Nieto et al. [37] showed that any suitable numerical method for ordinary differential equations can be applied to solve numerically fuzzy differential equations under generalized differentiability, and also they implemented the generalized Euler approximation method for solving first order linear fuzzy differential equations. Allahviranloo et al. [11] have been used the concept of generalised differentiability and applied differential transformation method for solving fuzzy differential equations. In 2011, Khasan et al. [35] have been studied first order linear fuzzy differential equations by using the generalized differentiability concept and presented the general form of their solutions. Recently, Ghanbari et al. [26] have been considered Seikkala’s derivative and applied a numerical algorithm for solving first order fuzzy differential equation, based on extended Runge-Kutta-like formulae of order 4.

In this work, the first order linear fuzzy differential equation is solved via Variational Iteration Method (VIM) and Adomian Decomposition Method (ADM). We replace the initial problem by its parametric form and then solve the new system which consist of two classic ordinary differential equations with initial conditions, then check to see whether this solution define a fuzzy function.

The structure of this paper is organized as follows. In Section 2, some basic definitions which will be used later in the paper are provided. In Section 3, we present the different parametric forms of a fuzzy differential equation by using the strongly generalized differentiability concept. In Section 4, we state the basic concepts of VIM and ADM, and apply these two methods on parametric forms of a fuzzy differential equation. In Section 5, we give two numerical examples. We conclude in Section 6.

2 Preliminaries

Definition 2.1. A fuzzy number is a function \( u : \mathbb{R} \rightarrow [0,1] \) satisfying the following properties:

(i) \( u \) is normal, i.e. \( \exists x_0 \in \mathbb{R} \) with \( u(x_0) = 1 \),

(ii) \( u \) is a convex fuzzy set,

(iii) \( u \) is upper semi-continuous on \( \mathbb{R} \),

(iv) \( \{ x \in \mathbb{R} : u(x) > 0 \} \) is compact, where \( \overline{A} \) denotes the closure of \( A \).

The set of all these fuzzy numbers is denoted by \( E \). Obviously, \( \mathbb{R} \subset E \). Here \( \mathbb{R} \subset E \) is understood as \( \mathbb{R} = \{ \chi_{\{x\}} : x \) is usual real number\}. For \( 0 < r \leq 1 \), denote \( [u]_r = \{ x \in \mathbb{R} : u(x) \geq r \} \) and \( [u]_0 = \{ x \in \mathbb{R} : u(x) > 0 \} \). It is well-known that for each \( r \in [0,1] \), \( [u]_r \) is a
An arbitrary fuzzy number in the parametric form is represented by an ordered pair of functions \((\underline{u}(r), \overline{u}(r))\), \(0 \leq r \leq 1\), which satisfy the following requirements.

(i) \(\underline{u}(r)\) is a bounded left continuous non-decreasing function over \([0, 1]\).

(ii) \(\overline{u}(r)\) is a bounded left continuous non-increasing function over \([0, 1]\).

(iii) \(\underline{u}(r) \leq \overline{u}(r)\), \(0 \leq r \leq 1\).

A crisp number \(\alpha\) is simply represented by \(\underline{u}(r) = \overline{u}(r) = \alpha\), \(0 \leq r \leq 1\).

We recall that for arbitrary fuzzy numbers \(u = (\underline{u}(r), \overline{u}(r))\) and real number \(k\),

(a) \(u = v\) if and only if \(\underline{u}(r) = \overline{v}(r)\) and \(\overline{u}(r) = \overline{v}(r)\).

(b) \(u \oplus v = (\underline{u} \oplus \overline{v}, \overline{u} \oplus \overline{v}) = (\underline{u}(r) + \overline{v}(r), \overline{u}(r) + \overline{v}(r))\).

(c) 
\[
k \odot u = \begin{cases} 
(k \odot \underline{u}, k \odot \overline{u}) = (k \underline{u}(r), k \overline{u}(r)), & k > 0; \\
(k \odot \underline{u}, k \odot \overline{u}) = (k \overline{u}(r), k \underline{u}(r)), & k < 0.
\end{cases}
\]

The same as Eq. (2.1), we define
\[
D(u, v) = \sup_{0 \leq r \leq 1} \{\max[|\underline{u}(r) - \overline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|]\}.
\]

In this paper, we represent an arbitrary fuzzy number by a pair of functions \((\underline{u}(r), \overline{u}(r))\), \(0 \leq r \leq 1\).
Theorem 2.1. [12]

(i) If we define \( \tilde{0} = \chi_{\{0\}} \), then \( \tilde{0} \in E \) is a neutral element with respect to addition, i.e. 
\[ u \oplus \tilde{0} = \tilde{0} \oplus u = u, \text{ for all } u \in E. \]

(ii) With respect to \( \tilde{0} \), none of \( u \in E \setminus \mathbb{R} \) has inverse in \( E \) (with respect to \( \oplus \)).

(iii) For any \( a, b \in \mathbb{R} \) with \( a, b \geq 0 \) or \( a, b \leq 0 \) and any \( u \in E \), we have \( (a + b) \circ u = (a \circ u) \oplus (b \circ u) \). For general \( a, b \in \mathbb{R} \), the above property does not hold.

(iv) For any \( \lambda \in \mathbb{R} \) and any \( u, v \in E \), we have \( \lambda \circ (u \oplus v) = (\lambda \circ u) \oplus (\lambda \circ v) \).

(v) For any \( \lambda, \mu \in \mathbb{R} \) and any \( u \in E \), we have \( \lambda \circ (\mu \circ u) = (\lambda \cdot \mu) \circ u \).

Definition 2.3. Let \( E \) be a set of all fuzzy numbers, we say that \( f(x) \) is a fuzzy function if \( f : \mathbb{R} \rightarrow E \).

Definition 2.4. Consider \( u, v \in E \). If there exists \( w \in E \) such that \( u = v \oplus w \), then \( w \) is called the Hukuhara difference of \( u \) and \( v \) and it is denoted by \( u \sim_h v \).

In this paper the “\( \sim_h \)” sign stands always for Hukuhara difference and note that \( u \sim_h v \neq u \oplus (-1) \circ v \).

We recall that for \( a < b < c, a, b, c \in \mathbb{R} \), the triangular fuzzy number \( u = (a, b, c) \) determined by \( a, b, c \) is given such that \( \underline{u}(r) = a + (b - a)r \) and \( \overline{u}(r) = c - (c - b)r \) are the components of the parametric form of the triangular fuzzy number \( u \), where \( r \in [0, 1] \), i.e. \( u = (\underline{u}(r), \overline{u}(r)) \). Therefore, we can write \( u = (\underline{u}(0), u(1), \overline{u}(0)) \) where \( \underline{u}(1) = \overline{u}(1) = b \) and it is denoted by \( u(1), \underline{u}(0) = a \) and \( \overline{u}(0) = c \). The set of triangular fuzzy numbers will be denoted by \( E_r \). The following lemma gives a sufficient condition for the existence of the Hukuhara difference of two triangular fuzzy numbers.

Lemma 2.1. [15] Let \( u, v \in E_r \) be such that \( u(1) - \underline{u}(0) > 0 \), \( \overline{u}(0) - u(1) > 0 \) and \( \text{len}(v) = (\overline{u}(0) - \underline{u}(0)) \leq \min\{u(1) - \underline{u}(0), \overline{u}(0) - u(1)\} \). Then the Hukuhara difference \( u \sim_h v \) exists and \( u \sim_h v = (\underline{u}(0) - \underline{v}(0), u(1) - v(1), \overline{u}(0) - \overline{v}(0)) \).

Definition 2.5. [13] Let \( f : (a, b) \rightarrow E \) and \( x_0 \in (a, b) \). We say that \( f \) is strongly generalized differentiable on \( x_0 \) (Bede-Gal differentiability), if there exists an element \( f'(x_0) \in E \), such that

(i) for all \( h > 0 \) sufficiently small, \( \exists f(x_0 + h) \sim_h f(x_0), f(x_0) \sim_h f(x_0 - h) \) and the limits (in the metric \( D \))
\[
\lim_{h \searrow 0} \frac{f(x_0 + h) \sim_h f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) \sim_h f(x_0 - h)}{h} = f'(x_0),
\]
or

(ii) for all \( h > 0 \) sufficiently small, \( \exists f(x_0) \sim_h f(x_0 + h), f(x_0 - h) \sim_h f(x_0) \) and the limits
\[
\lim_{h \searrow 0} \frac{f(x_0) \sim_h f(x_0 + h)}{(-h)} = \lim_{h \searrow 0} \frac{f(x_0 - h) \sim_h f(x_0)}{(-h)} = f'(x_0),
\]
(iii) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \sim_h f(x_0)$, $f(x_0 - h) \sim_h f(x_0)$ and the limits
\[
\lim_{h \downarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \downarrow 0} \frac{f(x_0 - h) - f(x_0)}{-h} = f'(x_0),
\]

or

(iv) for all $h > 0$ sufficiently small, $\exists f(x_0) \sim_h f(x_0 + h)$, $f(x_0) \sim_h f(x_0 - h)$ and the limits
\[
\lim_{h \downarrow 0} \frac{f(x_0) - f(x_0 + h)}{-h} = \lim_{h \downarrow 0} \frac{f(x_0) - f(x_0 - h)}{h} = f'(x_0),
\]

$h$ and $(-h)$ at denominators mean $\frac{1}{h}$ and $-\frac{1}{h}$, respectively.

**Theorem 2.2.** [19] Let $f : \mathbb{R} \to E$ be a fuzzy function and denote $f(t) = (\underline{f}(t;r), \overline{f}(t;r))$, for each $r \in [0, 1]$. Then

1. If $f$ is differentiable in the first form (i), then $\underline{f}(t;r)$ and $\overline{f}(t;r)$ are differentiable functions and $f'(t) = (\underline{f}'(t;r), \overline{f}'(t;r))$.

2. If $f$ is differentiable in the second form (ii), then $\underline{f}(t;r)$ and $\overline{f}(t;r)$ are differentiable functions and $f'(t) = (\overline{f}'(t;r), \underline{f}'(t;r))$.

**Theorem 2.3.** [15] Let $f : (a, b) \to E$ be strongly generalized differentiable on each point $x \in (a, b)$ in the sense of Definition 2.5(iii) or 2.5(iv). Then $f'(x) \in \mathbb{R}$ for all $x \in (a, b)$.

### 3 First order linear fuzzy differential equations

In this paper, we consider the fuzzy initial-value problem
\[
\begin{cases}
  y'(x) = (a(x) \odot y(x)) \oplus b(x), \\
  y(x_0) = y_0,
\end{cases}
\]
where $a : (x_0, \infty) \to \mathbb{R}$, $y_0 \in E_r$ and $b : (x_0, \infty) \to E_r$. This problem is not equivalent to the following other two problems:
\[
\begin{cases}
  y'(x) \oplus ((-a(x)) \odot y(x)) = b(x), \\
  y(x_0) = y_0,
\end{cases}
\]
and
\[
\begin{cases}
  y'(x) \oplus (-b(x)) = a(x) \odot y(x), \\
  y(x_0) = y_0.
\end{cases}
\]
Also, in the general case, problems (3.3) and (3.4) are not equivalent. The following theorem give me solutions of Eqs. (3.2)-(3.4).
Theorem 3.1. [14] Let
\[ y_1(x) = e^{\int_{x_0}^{x} a(t) dt} \left( y_0 \oplus \int_{x_0}^{x} b(t) \odot e^{-\int_{x_0}^{t} a(s) ds} dt \right), \]
and
\[ y_2(x) = e^{\int_{x_0}^{x} a(t) dt} \left( y_0 \odot h \int_{x_0}^{x} (-b(t)) \odot e^{-\int_{x_0}^{0} a(s) ds} dt \right), \]
provided that the Hukuhara difference \( y_0 \odot h \int_{x_0}^{x} (-b(t)) \odot e^{-\int_{x_0}^{0} a(s) ds} dt \) exists.
(a) If \( a(x) > 0 \) for \( x \in (x_0, x_1) \) then \( y_1 \) is \((i)\)-differentiable and it is a solution of the problem (3.2) in the \((x_0, x_1)\) interval.
(b) If \( a(x) < 0 \) for \( x \in (x_0, x_1) \) and if the Hukuhara difference \( y_0 \odot h \int_{x_0}^{x} (-b(t)) \odot e^{-\int_{x_0}^{0} a(s) ds} dt \) exists for \( x \in (x_0, x_1) \) then \( y_2 \) is \((ii)\)-differentiable and it is a solution of the problem (3.2) in the \((x_0, x_1)\) interval.
(c) If \( a(x) < 0 \) for \( x \in (x_0, x_1) \) and if \( y_1(x+h) \sim_h y_1(x) \) and \( y_1(x-h) \sim_h y_1(x) \) exist or \( y_1(x+h) \sim_h y_1(x-h) \) and \( y_1(x-h) \sim_h y_1(x) \) exist for \( h \) sufficiently small and \( x \in (x_0, x_1) \), then \( y_1 \) is a solution of (3.3) or (3.4) respectively.
(d) If \( a(x) > 0 \) for \( x \in (x_0, x_1) \), the Hukuhara difference \( y_0 \odot h \int_{x_0}^{x} (-b(t)) \odot e^{-\int_{x_0}^{0} a(s) ds} dt \) exists for \( x \in (x_0, x_1) \) and if \( y_2(x+h) \sim_h y_2(x) \) and \( y_2(x-h) \sim_h y_2(x) \) exist or \( y_2(x+h) \sim_h y_2(x-h) \) and \( y_2(x-h) \sim_h y_2(x) \) exist for \( h \) sufficiently small and \( x \in (x_0, x_1) \), then \( y_2 \) is a solution of (3.3) or (3.4) respectively.

Remark 3.1. The solutions provided by Theorem 3.1, may have a decreasing length of their support. In this case, if for example \( a(x) < 0 \) for any \( x \in (x_0, \infty) \) and \( y_0 \sim_h \int_{x_0}^{x} (-b(t)) \odot e^{-\int_{x_0}^{0} a(s) ds} dt \) exists for any \( x \in (x_0, \infty) \) and if it is bounded, then \( \lim_{x \to \infty} y_2(x) = 0 \), that is, the solution \( y_2 \) is asymptotically stable (see Example 5.2).

According to Theorems 2.2 and 3.1 and parametric form of a fuzzy number, if \( a(x) > 0 \) then we may replace Eq. (3.2) by the equivalent system
\[
\begin{align*}
\frac{d}{dr} y'(x;r) &= a(x) \cdot y(x;r) + b(x;r), & y(x_0;r) = y_0(r), \\
\frac{d}{dr} y(x;r) &= a(x) \cdot y(x;r) + b(x;r), & y(x_0;r) = y_0(r),
\end{align*}
\]
and if \( a(x) < 0 \) then we may replace Eq. (3.2) by the equivalent system
\[
\begin{align*}
\frac{d}{dr} y'(x;r) &= a(x) \cdot y(x;r) + b(x;r), & y(x_0;r) = y_0(r), \\
\frac{d}{dr} y(x;r) &= a(x) \cdot y(x;r) + b(x;r), & y(x_0;r) = y_0(r).
\end{align*}
\]
For every prefixed \( r \in [0, 1] \), the above systems represent ordinary initial value problems for which any converging classical numerical procedure can be applied.

4 The methods

In what follows we will highlight briefly the main points of the variational iteration method and the Adomian decomposition method.
4.1 Variational iteration method (VIM)

The VIM is proposed by He [30, 31] as a modification of a general Lagrange multiplier method [32]. This method has been shown to solve effectively, easily, and accurately a large class of linear and nonlinear problems with approximations converging rapidly to accurate solutions [5, 6, 41]. To illustrate its basic idea of the technique, we consider the following general nonlinear system:

\[ L[u(x)] + N[u(x)] = g(x), \]  

(4.7)

where \( L \) is a linear operator, \( N \) is a nonlinear operator, and \( g(x) \) is a given continuous function. The basic character of the method is to construct a correction functional for system (4.7), which reads

\[ u_{n+1}(x) = u_n(x) + \int_{x_0}^x \lambda(\tau) \{ L[u_n(\tau)] + N[\tilde{u}_n(\tau)] - g(\tau) \} d\tau, \]

where \( \lambda(\tau) \) is a general Lagrange multiplier [30, 31, 32] which can be identified optimally via variational theory, the subscript \( n \) denotes the \( n \)-th order approximation and \( \tilde{u}_n \) is considered as a restricted variation [24], i.e. \( \delta \tilde{u}_n = 0 \). For linear problems, its exact solution can be obtained by only one iteration step due to the fact the Lagrange multiplier can be exactly identified.

For solving Eqs. (3.5) by VIM, we construct the following correction functionals

\[
\begin{aligned}
    \frac{y_{n+1}(x; r)}{y_n(x; r)} &= \frac{y_n(x; r) + \int_{x_0}^x \lambda_1(\tau) \{ y_n'(\tau; r) - a(\tau) \cdot y_n(\tau; r) - b(\tau; r) \} d\tau}{\frac{y_n(x; r) + \int_{x_0}^x \lambda_2(\tau) \{ y_n'(\tau; r) - a(\tau) \cdot y_n(\tau; r) - b(\tau; r) \} d\tau}{y_n(x; r)}}, \\
    \frac{\bar{y}_{n+1}(x; r)}{\bar{y}_n(x; r)} &= \frac{\bar{y}_n(x; r) + \int_{x_0}^x \lambda_1(\tau) \{ \bar{y}_n'(\tau; r) - a(\tau) \cdot \bar{y}_n(\tau; r) - \bar{b}(\tau; r) \} d\tau}{\frac{\bar{y}_n(x; r) + \int_{x_0}^x \lambda_2(\tau) \{ \bar{y}_n'(\tau; r) - a(\tau) \cdot \bar{y}_n(\tau; r) - \bar{b}(\tau; r) \} d\tau}{\bar{y}_n(x; r)}},
\end{aligned}
\]

(4.8)

Calculating variation with respect to \( \frac{y_n}{y_n} \) and \( \frac{\bar{y}_n}{\bar{y}_n} \), noticing that \( \delta y_n(x_0) = 0 \) and \( \delta \bar{y}_n(x_0) = 0 \), yields

\[ \delta y_{n+1}(x; r) = \delta y_n(x; r) + \delta \int_{x_0}^x \lambda_1(\tau) \{ y_n'(\tau; r) - a(\tau) \cdot y_n(\tau; r) - b(\tau; r) \} d\tau = (1 + \lambda_1(\tau)) \delta y_n(\tau; r)|_{x_0} - \int_{x_0}^x \lambda_1(\tau) \delta y_n(\tau; r) d\tau = 0, \]

and

\[ \delta \bar{y}_{n+1}(x; r) = \delta \bar{y}_n(x; r) + \delta \int_{x_0}^x \lambda_2(\tau) \{ \bar{y}_n'(\tau; r) - a(\tau) \cdot \bar{y}_n(\tau; r) - \bar{b}(\tau; r) \} d\tau = (1 + \lambda_2(\tau)) \delta \bar{y}_n(\tau; r)|_{x_0} - \int_{x_0}^x \lambda_2(\tau) \delta \bar{y}_n(\tau; r) d\tau = 0. \]

Therefore, we have the following stationary conditions:

\[ \lambda_1'(\tau) + a(\tau) \cdot \lambda_1(\tau) = 0, \quad 1 + \lambda_i(\tau)|_{x_0} = 0, \quad i = 1, 2. \]

So, the Lagrange multipliers can be readily identified

\[ \lambda_1(\tau) = \lambda_2(\tau) = -e^{\int_{x_0}^\tau a(s) ds}. \]
Substituting these values of the Lagrange multipliers into functionals (4.8) gives the iteration formulas:

\[
\begin{align*}
\bar{y}_{n+1}(x;r) &= \bar{y}_n(x;r) - \int_{x_0}^x e^{\int_{\tau}^{x} a(s)ds} \{ \bar{y}_n'(\tau;r) - a(\tau) \cdot \bar{y}_n(\tau;r) - \bar{b}(\tau;r) \} d\tau, \\
\bar{y}_n(x;r) &= \frac{y(\tau)}{\bar{y}_n(\tau;r)} - \int_{x_0}^x e^{\int_{\tau}^{x} a(s)ds} \{ \bar{y}_n'(\tau;r) - a(\tau) \cdot \bar{y}_n(\tau;r) - \bar{b}(\tau;r) \} d\tau,
\end{align*}
\]  
(4.9)

the values of \( \bar{y}_0(x;r) \) and \( \bar{y}_0(x;r) \) are initial approximations and chosen as follows:

\[
\bar{y}_0(x;r) = y(x_0;r) = y_0(r), \quad \bar{y}_0(x;r) = \bar{y}(x_0;r) = \bar{y}_0(r).
\]

Iteration formulas (4.9) whichever will give several approximations, and the exact solution is obtained at the limit of the resulting successive approximations, i.e.,

\[
\bar{y}(x;r) = \lim_{n \to \infty} \bar{y}_n(x;r), \quad \bar{y}(x;r) = \lim_{n \to \infty} \bar{y}_n(x;r),
\]

and \( y(x) = (y(x;r), \bar{y}(x;r)) \) is exact solution. The proof of convergence of variational iteration method, is given in [42].

Similarly, the iteration formulas for Eq. (3.6) will obtain as follows:

\[
\begin{align*}
\bar{y}_{n+1}(x;r) &= \bar{y}_n(x;r) - \int_{x_0}^x e^{\int_{\tau}^{x} a(s)ds} \{ \bar{y}_n'(\tau;r) - a(\tau) \cdot \bar{y}_n(\tau;r) - \bar{b}(\tau;r) \} d\tau, \\
\bar{y}_n(x;r) &= y(x_0;r) = y_0(r), \quad \bar{y}_0(x;r) = \bar{y}(x_0;r) = \bar{y}_0(r).
\end{align*}
\]
(4.10)

where

\[
\bar{y}_0(x;r) = y(x_0;r) = y_0(r), \quad \bar{y}_0(x;r) = \bar{y}(x_0;r) = \bar{y}_0(r).
\]

Iteration formulas (4.10) whichever will give several approximations, and the exact solution is obtained at the limit of the resulting successive approximations, i.e.,

\[
\bar{y}(x;r) = \lim_{n \to \infty} \bar{y}_n(x;r), \quad \bar{y}(x;r) = \lim_{n \to \infty} \bar{y}_n(x;r),
\]

and \( y(x) = (y(x;r), \bar{y}(x;r)) \) is exact solution.

### 4.2 Adomian decomposition method (ADM)

Adomian decomposition method [7, 8] defines the unknown functions \( u(t) \) by an infinite series

\[
u(t) = \sum_{n=0}^{\infty} u_n(t),
\]
(4.11)

where the components \( u_n(t) \) are usually determined recurrently. The nonlinear operator \( N(u) \) can be decomposed into an infinite series of polynomials given by

\[
N(u) = \sum_{n=0}^{\infty} A_n,
\]

where \( A_n \) are the so-call Adomian polynomials of \( u_0, u_1, \ldots, u_n \) defined by

\[
A_n = \frac{1}{n!} \left. \frac{d^n}{d\lambda^n} \left[ N \left( \sum_{i=0}^{\lambda} \lambda^i u_i \right) \right] \right|_{\lambda=0}, \quad n = 0, 1, 2, \ldots.
\]
Practical formula for the calculation of Adomian polynomials are given in [21, 28]. For later numerical computation, we let the expression

$$\Phi_n(t) = \sum_{k=0}^{n} u_k(t),$$

to denote the \(n\)-term approximation to \(u(t)\).

We rewrite Eqs. (3.5) in the following form:

$$\begin{align*}
\frac{y}{y}(x; r) &= y_0(r) + \int_{x_0}^{x} a(s) \cdot \frac{y}{y}(s; r) \, ds + \int_{x_0}^{x} b(s; r) \, ds, \\
\frac{\psi}{\psi}(x; r) &= \psi_0(r) + \int_{x_0}^{x} a(s) \cdot \frac{\psi}{\psi}(s; r) \, ds + \int_{x_0}^{x} b(s; r) \, ds.
\end{align*}$$

(4.12)

To use ADM, let

$$\begin{align*}
y_n(x; r) &= \sum_{n=0}^{\infty} y(x; r), \\
\psi_n(x; r) &= \sum_{n=0}^{\infty} \psi(x; r),
\end{align*}$$

(4.13)

substituting (4.13) in (4.12) we have:

$$\begin{align*}
\sum_{n=0}^{\infty} y_n(x; r) &= y_0(r) + \int_{x_0}^{x} b(s; r) \, ds + \sum_{n=0}^{\infty}(\int_{x_0}^{x} a(s) \cdot y_n(s; r) \, ds), \\
\sum_{n=0}^{\infty} \psi_n(x; r) &= \psi_0(r) + \int_{x_0}^{x} b(s; r) \, ds + \sum_{n=0}^{\infty}(\int_{x_0}^{x} a(s) \cdot \psi_n(s; r) \, ds).
\end{align*}$$

Identifying the zeroth components

$$\begin{align*}
y_0(x; r) &= y_0(r) + \int_{x_0}^{x} b(s; r) \, ds, \\
\psi_0(x; r) &= \psi_0(r) + \int_{x_0}^{x} b(s; r) \, ds,
\end{align*}$$

the remaining components \(y_n(x; r)\) and \(\psi_n(x; r), n \geq 1\), can be determined by using the recurrence relations

$$\begin{align*}
y_{n+1}(x; r) &= \int_{x_0}^{x} a(s) \cdot y_n(s; r) \, ds, \quad n \geq 0, \\
\psi_{n+1}(x; r) &= \int_{x_0}^{x} a(s) \cdot \psi_n(s; r) \, ds, \quad n \geq 0.
\end{align*}$$

(4.14)

By Eqs. (4.14), we approximate \(y(x; r)\) with

$$\Phi_n(x) = \sum_{k=0}^{n} y_k(x; r),$$

and approximate \(\psi(x; r)\) with

$$\Phi_n(x) = \sum_{k=0}^{n} \psi_k(x; r),$$

and the exact solution is obtained at the limit of the resulting approximations, i.e.,

$$\begin{align*}
y(x; r) &= \lim_{n \to \infty} \Phi_n(x; r), \\
\psi(x; r) &= \lim_{n \to \infty} \Phi_n(x; r),
\end{align*}$$

and \(y(x) = (y(x; r), \psi(x; r))\) is exact solution. The proofs of convergence of Adomian decomposition method are given in [1, 2].
Similarly, the recurrence relations for Eqs. (3.6) are obtained as follows:

\[
\begin{align*}
\overline{y}_{n+1}(x; r) &= \int_{x_0}^{x} a(s) \cdot \overline{y}_n(s; r) \, ds, \quad n \geq 0, \\
y_{n+1}(x; r) &= \int_{x_0}^{x} a(s) \cdot y_n(s; r) \, ds, \quad n \geq 0,
\end{align*}
\]

where

\[
\overline{y}_0(x; r) = \overline{y}_0(r) + \int_{x_0}^{x} b(s; r) \, ds, \quad y_0(x; r) = y_0(r) + \int_{x_0}^{x} \overline{b}(s; r) \, ds.
\]

By Eq. (4.15), we approximate \( \overline{y}(x; r) \) with

\[
\Phi_n(x) = \sum_{k=0}^{n} \overline{y}_k(x; r),
\]

and approximate \( y(x; r) \) with

\[
\Phi_n(x) = \sum_{k=0}^{n} y_k(x; r),
\]

and the exact solution is obtained at the limit of the resulting approximations, i.e.,

\[
\overline{y}(x; r) = \lim_{n \to \infty} \Phi_n(x; r), \quad y(x; r) = \lim_{n \to \infty} \Phi_n(x; r),
\]

and \( y(x) = (\overline{y}(x; r), \overline{y}(x; r)) \) is exact solution.

5 Numerical results

In this section, we apply VIM and ADM to two examples. We use MATLAB software in all the calculations done in this section.

Example 5.1. Consider the following fuzzy initial-value problem

\[
\begin{align*}
y'(x) &= (x^2 \odot y(x)) \oplus ((1, 2, 3) \odot x^2), \quad x \geq 0, \\
y(0) &= (1, 2, 3).
\end{align*}
\]

By Theorem 3.1(a), we get

\[
y(x) = (2e^{1/3x^3} - 1, 4e^{1/3x^3} - 2, 6e^{1/3x^3} - 3),
\]

is solution of (5.16) on \((0, \infty)\).

By Definition 2.2, we can rewrite Eqs. (5.16) and (5.17) in the form:

\[
\begin{align*}
y'(x) &= (x^2 \odot y(x)) \oplus (x^2 \odot (1 + r), x^2 \odot (3 - r)), \quad x \geq 0, \\
y(0) &= (1 + r, 3 - r),
\end{align*}
\]
and
\[ y(x) = (y(x; r), \varphi(x; r)) = \left( (2e^{1/3x^3} - 1)(1 + r), (2e^{1/3x^3} - 1)(3 - r) \right), \] (5.19)

respectively. Note that:
\[ y(x; 0) = 2e^{1/3x^3} - 1, \] (5.20)
\[ y(x; 1) = \varphi(x; 1) = 4e^{1/3x^3} - 2, \] (5.21)
\[ \varphi(x; 0) = 6e^{1/3x^3} - 3. \] (5.22)

According to Theorems 2.2 and 3.1 and parametric form of a fuzzy number, since \( a(x) = x^2 > 0 \) on \( x \in (0, \infty) \) then we replace Eq. (5.18) by the following equivalent system:
\[
\begin{cases}
  y'(x; r) = x^2 \cdot y(x; r) + x^2 \cdot (1 + r), & y(0; r) = 1 + r, \\
  \varphi'(x; r) = x^2 \cdot \varphi(x; r) + x^2 \cdot (3 - r), & \varphi(0; r) = (3 - r).
\end{cases}
\] (5.23)

Now, we solve Eq. (5.23) via VIM and ADM.

- **Using variational iteration method (VIM)**

According to Eq. (4.9), we can obtain the following iteration formulas:
\[
\begin{aligned}
  y_{n+1}(x; r) &= y_n(x; r) - \int_0^x e^{1/3(x^3 - r^3)} \{ y'_n(\tau; r) - \tau^2 \cdot y_n(\tau; r) - \tau^2 (1 + r) \} \, d\tau, \\
  \varphi_{n+1}(x; r) &= \varphi_n(x; r) - \int_0^x e^{1/3(x^3 - r^3)} \{ \varphi'_n(\tau; r) - \tau^2 \cdot \varphi_n(\tau; r) - \tau^2 (3 - r) \} \, d\tau.
\end{aligned}
\]

We start with initial approximations
\[ y_0(x; r) = y(0; r) = 1 + r, \quad \varphi_0(x; r) = \varphi(0; r) = 3 - r, \]
and by the above iteration formulas, we can obtain
\[ y_1(x; r) = (2e^{1/3x^3} - 1)(1 + r), \]
\[ \varphi_1(x; r) = (2e^{1/3x^3} - 1)(3 - r), \]
\[ y_2(x; r) = (2e^{1/3x^3} - 1)(1 + r), \]
\[ \varphi_2(x; r) = (2e^{1/3x^3} - 1)(3 - r), \]
\[ \vdots \]

which are exactly the same as components of Eq. (5.19). Therefore, by only one iteration, the exact solution is obtained.
Using Adomian decomposition method (ADM)

According to Eqs. (4.14), we have

\[
\begin{aligned}
\frac{y_{n+1}(x; r)}{y_n(x; r)} &= \int_0^x s^2 \cdot \frac{y_n(s; r)}{y_n(s; r)} \, ds, \quad n \geq 0, \\
\frac{\overline{y}_{n+1}(x; r)}{\overline{y}_n(x; r)} &= \int_0^x s^2 \cdot \overline{y}_n(s; r) \, ds, \quad n \geq 0,
\end{aligned}
\]

where

\[
y_0(x; r) = \frac{1}{3} x^3 (1 + r) + (1 + r), \quad \overline{y}_0(x; r) = \frac{1}{3} x^3 (3 - r) + (3 - r),
\]

We approximate \( y(x; r) \) and \( \overline{y}(x; r) \), with \( \Phi_6(x; r) \) and \( \overline{\Phi}_6(x; r) \), respectively, as follows:

\[
\begin{aligned}
\Phi_6(x; r) &= \sum_{i=0}^{6} y_i(x; r) \\
&= \left( 1 + \frac{2 x^3}{3} + \frac{x^6}{9} + \frac{x^9}{81} + \frac{x^{12}}{972} + \frac{x^{15}}{14580} + \frac{x^{18}}{262440} + \frac{x^{21}}{11022480} \right) (1 + r),
\end{aligned}
\]

\[
\begin{aligned}
\overline{\Phi}_6(x; r) &= \sum_{i=0}^{6} \overline{y}_i(x; r) \\
&= \left( 1 + \frac{2 x^3}{3} + \frac{x^6}{9} + \frac{x^9}{81} + \frac{x^{12}}{972} + \frac{x^{15}}{14580} + \frac{x^{18}}{262440} + \frac{x^{21}}{11022480} \right) (3 - r).
\end{aligned}
\]

Therefore, we have

\[
\begin{aligned}
\Phi_6(x; 0) &= 1 + \frac{2 x^3}{3} + \frac{x^6}{9} + \frac{x^9}{81} + \frac{x^{12}}{972} + \frac{x^{15}}{14580} + \frac{x^{18}}{262440} + \frac{x^{21}}{11022480}, \\
\Phi_6(x; 1) &= 2 + \frac{4 x^3}{3} + \frac{2 x^6}{9} + \frac{x^9}{81} + \frac{x^{12}}{486} + \frac{x^{15}}{7290} + \frac{x^{18}}{131220} + \frac{x^{21}}{5511240}, \\
\overline{\Phi}_6(x; 0) &= 3 + 2 x^3 + \frac{x^6}{3} + \frac{x^9}{27} + \frac{x^{12}}{324} + \frac{x^{15}}{4860} + \frac{x^{18}}{87480} + \frac{x^{21}}{3674160},
\end{aligned}
\]

which are approximations for Eqs. (5.20)-(5.22), respectively. We see that by VIM, the exact solution is obtained by only one iteration. But by ADM, we can obtain only an approximate solution. The results obtained by 6-term of ADM (Eqs. (5.24)-(5.26)) are compared with exact solutions (Eqs. (5.20)-(5.22)) in Fig. 1.
Example 5.2. Consider the following fuzzy initial-value problem

\[
\begin{cases}
y'(x) = ((-x) \odot y(x)) \oplus (1, 3, 4) \odot x \cdot e^{-x^2}, & x \geq 0, \\
y(0) = (1, 5, 9).
\end{cases}
\] (5.27)

In this case we have:

\[
\int_0^x ((-1, 3, 4)t^2 \cdot e^{-t^2})e^{1/2t^2} dt = \left(4(e^{-1/2x^2} - 1), 3(e^{-1/2x^2} - 1), (e^{-1/2x^2} - 1)\right),
\]

and

\[
\text{len} \left( \int_0^x ((-1, 3, 4)t^2 \cdot e^{-t^2})e^{1/2t^2} dt \right) = 3(e^{-1/2x^2} - 1) \leq \min\{5 - 1, 9 - 4\} = 4.
\]

Then by Lemma 2.1 the Hukuhara difference \( y_0 \sim_h \int_0^x ((-1, 3, 4)t^2 \cdot e^{-t^2})e^{1/2t^2} dt \) exists for any \( x \in (0, \infty) \) and since \( a(x) = -x < 0 \) for any \( x \in (0, \infty) \), therefore, by Theorem 3.1(b) we get that

\[
y(x) = (5e^{-1/2x^2} - 4e^{-x^2}, 8e^{-1/2x^2} - 3e^{-x^2}, 10e^{-1/2x^2} - e^{-x^2}),
\] (5.28)

is solution of (5.27) on \((0, \infty)\).

According to Remark 3.1, we observe that asymptotically, the uncertainty disappears on the fuzzy system. We could say that the solution is “asymptotically certain” (see [14]). The graphical representation of the solution can be seen in Fig. 2.
As before, we can rewrite Eqs. (5.27) and (5.28) in the form:

\[
\begin{align*}
\begin{cases}
y'(x) = ((-x) \odot y(x)) \oplus (1 + 2r)x e^{-x^2}, (4 - r)x \cdot e^{-x^2}, & x \geq 0, \\
y(0) = (1 + 4r; 9 - 4r),
\end{cases}
\end{align*}
\]  (5.29)

and

\[
\begin{align*}
y(x) &= \left(\overline{y}(x; r), \underline{y}(x; r)\right) \\
&= \left(e^{-1/2x^2}(5 + 3r) + e^{-x^2}(r - 4), 2e^{-1/2x^2}(5 - r) - e^{-x^2}(1 + 2r)\right),
\end{align*}
\]  (5.30)

respectively. Note that:

\[
\begin{align*}
\overline{y}(x; 0) &= 5e^{-1/2x^2} - 4e^{-x^2}, \\
\underline{y}(x; 1) &= \overline{y}(x; 1) = 8e^{-1/2x^2} - 3e^{-x^2}, \\
\underline{y}(x; 0) &= 10e^{-1/2x^2} - e^{-x^2}.
\end{align*}
\]  (5.31)

According to Theorems 2.2 and 3.1 and parametric form of a fuzzy number, since \(a(x) = -x < 0\) on \(x \in (0, \infty)\) then we replace Eq. (5.29) by the following equivalent system:

\[
\begin{align*}
\begin{cases}
\overline{y}'(x; r) = (-x) \cdot \overline{y}(x; r) + (1 + 2r)x \cdot e^{-x^2}, & \overline{y}(0; r) = (9 - 4r), \\
\underline{y}'(x; r) = (-x) \cdot \underline{y}(x; r) + (4 - r)x \cdot e^{-x^2}, & \underline{y}(0; r) = (1 + 4r).
\end{cases}
\end{align*}
\]  (5.34)

Now, we solve Eqs. (5.34) via VIM and ADM.
**Using variational iteration method (VIM)**

According to Eqs. (4.10), we can obtain the following iteration formulas:

\[
\begin{align*}
\overline{y}_{n+1}(x; r) &= \overline{y}_n(x; r) - \int_0^x e^{1/2(r^2-x^2)} \{ \overline{y}_n(\tau; r) + \tau \cdot \overline{y}_n(\tau; r) - (1 + 2r)\tau \cdot e^{-r^2} \} d\tau, \\
y_{n+1}(x; r) &= y_n(x; r) - \int_0^x e^{1/2(r^2-x^2)} \{ y_n'(\tau; r) + \tau \cdot y_n'(\tau; r) - (4 - r)\tau \cdot e^{-r^2} \} d\tau,
\end{align*}
\]

We start with initial approximations

\[
y_0(x; r) = \overline{y}(0; r) = (9 - 4r), \quad y_0'(t; r) = y(0; r) = (1 + 4r).
\]

and by the above iteration formulas, we can obtain

\[
y_1(x; r) = 2e^{-1/2x^2}(5 - r) - e^{-x^2} (1 + 2r), \\
y_1'(x; r) = e^{-1/2x^2}(5 + 3r) + e^{-x^2} (r - 4), \\
y_2(x; r) = 2e^{-1/2x^2}(5 - r) - e^{-x^2} (1 + 2r), \\
y_2'(x; r) = e^{-1/2x^2}(5 + 3r) + e^{-x^2} (r - 4),
\]

which are exactly the same as components of Eq. (5.30). Therefore, by only one iteration, the exact solution is obtained.

**Using Adomian decomposition method (ADM)**

According to Eqs. (4.15), we have

\[
\begin{align*}
\overline{y}_{n+1}(x; r) &= -\int_0^x s \cdot \overline{y}_n(s; r) ds, \quad n \geq 0, \\
y_{n+1}(x; r) &= -\int_0^x s \cdot y_n(s; r) ds, \quad n \geq 0,
\end{align*}
\]

where

\[
y_0(x; r) = (9 - 4r) + (1 + 2r) \int_0^x s \cdot e^{-s^2} ds = (9 - 4r) + \frac{1}{2} (1 + 2r) (1 - e^{-x}), \\
y_0'(x; r) = (1 + 4r) + (4 - r) \int_0^x s \cdot e^{-s^2} ds = (1 + 4r) + \frac{1}{2} (4 - r) (1 - e^{-x}).
\]

We approximate \(\overline{y}(x; r)\) and \(y(x; r)\), with \(\overline{\Phi}_6(x; r)\) and \(\Phi_6(x; r)\), respectively, as follows:

\[
\overline{\Phi}_6(x; r) = \sum_{i=0}^{6} \overline{y}_i(x; r) = \frac{1}{128} (1279 - 258r) - \frac{1}{128} (636 - 130r)x^2 + \frac{1}{256} (319 - 66r)x^4 - \frac{1}{768} (159 - 34r)x^6 + \frac{1}{3072} (79 - 18r)x^8 - \frac{1}{15360} (39 - 10r)x^{10} + \frac{1}{92160} (19 - 6r)x^{12} - \frac{127}{128} (2r + 1)e^{-x^2},
\]
\[ \Phi_6(x; r) = \sum_{i=0}^{6} y_i(x; r) = \frac{1}{128}(385r + 636) - \frac{1}{128}(193r + 316)x^2 + \frac{1}{256}(97r + 156)x^4 - \frac{1}{768}(49r + 76)x^6 + \frac{1}{3072}(25r + 36)x^8 - \frac{1}{15360}(13r + 16)x^{10} + \frac{1}{92160}(7r + 6)x^{12} - \frac{127}{128}(4 - r)e^{-x^2}. \]

Therefore, we have

\[ \Phi_6(x; 0) = \frac{159}{32} - \frac{79}{32}x^2 + \frac{39}{64}x^4 - \frac{19}{192}x^6 + \frac{3}{256}x^8 - \frac{1}{960}x^{10} + \frac{1}{15360}x^{12} - \frac{127}{32}e^{-x^2}, \]

\[ \Phi_6(x; 1) = \Phi_6(x; 1) = \frac{1021}{128} - \frac{509}{128}x^2 + \frac{253}{256}x^4 - \frac{125}{768}x^6 + \frac{61}{3072}x^8 - \frac{29}{15360}x^{10} + \frac{13}{92160}x^{12} - \frac{381}{128}e^{-x^2}, \]

\[ \Phi_6(x; 0) = \frac{1279}{128} - \frac{639}{128}x^2 + \frac{319}{256}x^4 - \frac{53}{256}x^6 + \frac{79}{3072}x^8 - \frac{13}{5120}x^{10} + \frac{19}{92160}x^{12} - \frac{127}{128}e^{-x^2}, \]

which are approximations for Eqs. (5.31)-(5.33), respectively. We present in Figs. 3 and 4 the comparison of the exact solution and the approximate solution by 6-term and 25-term of ADM, respectively.

---

Fig. 3. Comparison of exact solution and approximate solution for Example 5.2, obtained by 6-term of ADM. Solid curve: exact solution; Dotted curve: approximate solution.
6 Conclusion

In this paper, we applied VIM and ADM for solving first order linear fuzzy differential equations. The original problem is replaced by two parametric ordinary differential equations which are then solved using the VIM and the ADM. For linear problems VIM give the exact solution by only one iteration. However, ADM provides the components of the exact solution, where these components should follow the summation give in Eq. (4.11). The exact solutions are compared with solutions obtained by means of the ADM. The results show that these two methods are useful for finding an accurate approximation of the exact solution, but, VIM is more effective than ADM and the convergence of VIM is much faster than ADM. Also, these two methods can be also used for solving $N$-th fuzzy differential equations.

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