Solving Fully Fuzzy Linear System of Equations in General Form

R. Ezzati 1*, S. Khezerloo 1, A. Yousefzadeh 1

(1) Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.

Abstract
In this work, we propose an approach for computing the positive solution of a fully fuzzy linear system where the coefficient matrix is a fuzzy $n \times n$ matrix. To do this, we use arithmetic operations on fuzzy numbers that introduced by Kaffman in [18] and convert the fully fuzzy linear system into two $n \times n$ and $2n \times 2n$ crisp linear systems. If the solutions of these linear systems don’t satisfy in positive fuzzy solution condition, we introduce the constrained least squares problem to obtain optimal fuzzy vector solution by applying the ranking function in given fully fuzzy linear system. Using our proposed method, the fully fuzzy linear system of equations always has a solution. Finally, we illustrate the efficiency of proposed method by solving some numerical examples.

Keywords: Fuzzy linear system; Fuzzy number; Ranking Function; Fuzzy number vector solution.

1 Introduction

There are many linear equation systems in many areas of science and engineering. According to Moore [20] exact numerical data might be unrealistic, but there could be considered uncertain data as more aspects of a real word problem. Fuzzy data is being used as a natural way to describe uncertain data. Fuzzy concept was introduced by Zadeh [23, 24]. So, we need to solve those linear systems in which all parameters , or some of them are fuzzy numbers. Friedman et al. [13, 14] applied an embedding method for solving $Ax = b$, where $A$ is a nonsingular crisp matrix. Moreover, they transformed the $n \times n$ fuzzy linear system to a $2n \times 2n$ real linear system. As a result, they solved the $2n \times 2n$ real linear system using the inverse matrix. There are many other numerical methods for solving

*Corresponding author. Email address: ezati@kiau.ac.ir, Tel: +989123618518
fuzzy linear systems such as Jacobi, Gauss-Seidel, Adomiam decomposition method and SOR iterative method [1, 2, 3, 4]. Cheng in [7] introduced a ranking function and Ghanbari et al. in [15] obtained fuzzy linear equation system by ranking functions so that the matrix \( A \) is non-fuzzy. Dehgan in [8, 9, 10] introduced full fuzzy system in which \( b \) and \( A \) are fuzzy vector and fuzzy matrix, respectively. Then Kumar in [19] obtained exact solution of fully fuzzy linear system by solving a linear programming. In this article, we will obtain positive exact solution or positive approximate solution of fully fuzzy linear system by using ranking function. This paper is organized as follows:

In Section 2, the basic concept of fuzzy number operation is brought. In Section 3, the main Section of the paper, a new approach based on ranking functions for solving fully fuzzy linear system, is suggested. The proposed idea is illustrated by solving some numerical examples in the Section 4. Finally conclusion is drawn in Section 5.

2 Preliminaries

In this section, we give some basic definitions of fuzzy numbers.

**Definition 2.1.** A fuzzy number is a fuzzy set \( \tilde{A} : R \rightarrow I = [0,1] \) which satisfies:

1. \( \tilde{A} \) is upper semi continuous.
2. \( \tilde{A}(x) = 0 \) outside some interval \([c,d]\).
3. There are real numbers \( a, b : c \leq a \leq b \leq d \) for which
   a. \( \tilde{A}(x) \) is monotonic increasing on \([c,a]\),
   b. \( \tilde{A}(x) \) is monotonic decreasing on \([b,d]\),
   c. \( \tilde{A}(x) = 1, \ a \leq x \leq b \).

The set of all fuzzy numbers (as given by Definition (2.1)) is denoted by \( F(\mathbb{R})^1 \). An alternative definition or parametric form of a fuzzy number which yields the same \( F(\mathbb{R})^1 \) is given by Kaleva [16].

**Definition 2.2.** A fuzzy number \( \tilde{A} \) is LR-type if there exit L (for left) and R (for right) and scalars \( \alpha > 0, \beta > 0 \) with

\[
\tilde{A}(x) = \begin{cases} 
L\left(\frac{x-a}{\alpha}\right), & x \leq a, \\
R\left(\frac{x-b}{\beta}\right), & x \geq a 
\end{cases} \tag{2.1}
\]

where \( L \) and \( R \) are strictly decreasing functions defined on \([0,1]\) and satisfy the conditions:

\[
L(0) = R(0) = 1, \quad L(1) = R(1) = 0, \quad 0 < L(x) < 1, \quad 0 < R(x) < 1, \quad x \neq 0, 1. \tag{2.2}
\]

The mean value of \( \tilde{A} \), \( m \), is a real number, and \( \alpha, \beta \) are called the left and right spreads, respectively. \( \tilde{A} \) is denoted by \((m,\alpha,\beta)_{LR}\).
Definition 2.3. $\tilde{M} = (m, \alpha, \beta)_{LR}$ is a triangular fuzzy number if $L = R = \max(0, 1 - x)$.

We denote the set of triangular fuzzy numbers by $F(\mathbb{R})^1_T$.

In this paper, we use ranking function $R(\tilde{x}) = \sqrt{x_0^2 + y_0^2}$ introduced by Cheng [7] which is based on centroid point where for any triangular fuzzy number $\tilde{x} = (m, \alpha, \beta)$, $x_0$ and $y_0$ are as follows:

\[
x_0 = m + \frac{1}{3}(\beta - \alpha),
\]
\[
y_0 = \frac{1}{3} \left( \frac{6m + (\beta - \alpha)}{4m + (\beta - \alpha)} \right).
\]

Remark 2.1. According to Eq. (2.1) and definition (2.3), throughout the paper, we assume that all the fuzzy numbers are triangular in the form $(m, \alpha, m + \beta)$.

Arithmetic operations between two triangular fuzzy numbers, defined on universal set of real numbers $\mathbb{R}$, are reviewed [18].

Theorem 2.1. [17], Let $\tilde{M} = (a, b, c)$ and $\tilde{N} = (x, y, z)$ are two arbitrary triangular fuzzy numbers and $\lambda > 0$ is a real number. Then

1. $\tilde{M} \oplus \tilde{N} = (a + x, b + y, c + z)$,
2. $-\tilde{M} = (-c, -b, -a)$,
3. $\tilde{M} \ominus \tilde{N} = (a - z, b - y, c - x)$,
4. Let $\tilde{M} = (a, b, c)$ be any triangular fuzzy number and $\tilde{N} = (x, y, z)$ be a non-negative triangular fuzzy number, then

\[
\tilde{M} \otimes \tilde{N} = \begin{cases} 
(ax, by, cz), & a \geq 0, \\
(az, by, cx), & a < 0, c \geq 0, \\
(az, by, cx), & c < 0.
\end{cases}
\]

Definition 2.4. A matrix $\tilde{A} = [\tilde{a}_{ij}]_{i,j=1}^n$ is called a fuzzy matrix if for all $i$ and $j$, $\tilde{a}_{ij} \in F(\mathbb{R})^1_T$. $\tilde{A}$ will be positive (negative) and denoted by $\tilde{A} > 0$ ($\tilde{A} < 0$) if for all $i$ and $j$, $\tilde{a}_{ij} > 0$ ($\tilde{a}_{ij} < 0$). Clearly, $\tilde{N} = (a, b, c)$ is positive (negative), if and only if $a > 0$ ($c < 0$). Non-negative and non-positive fuzzy matrices will be defined similarly.

Definition 2.5. A vector $\tilde{X} = (\tilde{x}_1, \ldots, \tilde{x}_n)^T$, denoted by $\tilde{X} \in F(\mathbb{R})^n_T$, is called a fuzzy numbers vector, where $\tilde{x}_i \in F(\mathbb{R})^1_T$, $i = 1, \ldots, n$.

3 Solutions of fully fuzzy linear system by ranking function

Definition 3.1. The $n \times n$ linear system

\[
\begin{align*}
(\tilde{a}_{11} \otimes \tilde{x}_1) + (\tilde{a}_{12} \otimes \tilde{x}_2) + \cdots + (\tilde{a}_{1n} \otimes \tilde{x}_n) &= \tilde{b}_1 \\
(\tilde{a}_{21} \otimes \tilde{x}_1) + (\tilde{a}_{22} \otimes \tilde{x}_2) + \cdots + (\tilde{a}_{2n} \otimes \tilde{x}_n) &= \tilde{b}_2 \\
&\vdots \\
(\tilde{a}_{n1} \otimes \tilde{x}_1) + (\tilde{a}_{n2} \otimes \tilde{x}_2) + \cdots + (\tilde{a}_{nn} \otimes \tilde{x}_n) &= \tilde{b}_n,
\end{align*}
\] (3.3)
or in its matrix form,
\[
\tilde{A} \otimes \tilde{x} = \tilde{b},
\] (3.4)
is called a fully fuzzy linear system of equations (FFLSE) where the coefficient matrix
\[
\tilde{A} = [\tilde{a}_{ij}]_{i,j=1}^n
\] is a fuzzy matrix and \( \tilde{b} = [\tilde{b}_1, \ldots, \tilde{b}_n]^T \) is a fuzzy number vector and the fuzzy number vector \( \tilde{x} \) is the unknown to be found.

**Definition 3.2.** We call \( \tilde{x} \in F(\mathbb{R})^n \) a solution of \( \tilde{A} \otimes \tilde{x} = \tilde{b} \) with respect to the ranking function \( R \) if and only if we have,
\[
\begin{cases}
By = b_2, \\
R(\tilde{a}_i^T \tilde{x}) = R(\tilde{b}_i),
\end{cases}
\]
where the crisp linear system \( By = b_2 \) is the 1-cut or mean value of system \( \tilde{A} \otimes \tilde{x} = \tilde{b} \) and \( \tilde{a}_i^T \) is the \( i \)-th row of \( A \).

**Notation 3.1.** Set
\[
\tilde{a}_{ij} = (a_{ij}, b_{ij}, c_{ij}), \quad \tilde{b}_i = (d_{i1}, d_{i2}, d_{i3}), \quad \tilde{x}_i = (x_i, y_i, z_i),
\]
\[
A = \begin{pmatrix} a_{ij} \end{pmatrix}_{n \times n}, \quad B = \begin{pmatrix} b_{ij} \end{pmatrix}_{n \times n}, \quad C = \begin{pmatrix} c_{ij} \end{pmatrix}_{n \times n},
\]
\[
d_1 = \begin{pmatrix} d_{i1} \end{pmatrix}_{n \times 1}, \quad d_2 = \begin{pmatrix} d_{i2} \end{pmatrix}_{n \times 1}, \quad d_3 = \begin{pmatrix} d_{i3} \end{pmatrix}_{n \times 1}.
\]

**Notation 3.2.** We break up the matrix \( A \) into two \( n \times n \) matrices such that their addition is \( A \). Let \( A^+ = \begin{pmatrix} a^+_{ij} \end{pmatrix}_{n \times n} \) and \( A^- = \begin{pmatrix} a^-_{ij} \end{pmatrix}_{n \times n} \), where,
\[
a^+_{ij} = \begin{cases} a_{ij} & a_{ij} \geq 0 \\ 0 & a_{ij} < 0 \end{cases}, \quad a^-_{ij} = \begin{cases} 0 & a_{ij} \geq 0 \\ a_{ij} & a_{ij} < 0 \end{cases}.
\]
Then \( A^+ + A^- = A \). We also break up the matrix \( C \) into two \( n \times n \) matrices, similarly.

Let \( C^+ = \begin{pmatrix} c^+_{ij} \end{pmatrix} \), and \( C^- = \begin{pmatrix} c^-_{ij} \end{pmatrix} \) be \( n \times n \) matrices, where,
\[
c^+_{ij} = \begin{cases} c_{ij} & c_{ij} \geq 0 \\ 0 & c_{ij} < 0 \end{cases}, \quad c^-_{ij} = \begin{cases} 0 & c_{ij} \geq 0 \\ c_{ij} & c_{ij} < 0 \end{cases}.
\]

**Theorem 3.1.** The system (3.4) with the multiplication defined in Theorem 2.1 is equivalent to
\[
\begin{pmatrix} A^+ & A^- \\ C^- & C^+ \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_3 \end{pmatrix},
\]
(3.5)
where \( \tilde{x}_j = (x_j, y_j, z_j), \ j = 1, \ldots, n, \) are nonnegative and \( x = (x_1, \ldots, x_n)^T, \ y = (y_1, \ldots, y_n)^T, \ z = (z_1, \ldots, z_n)^T. \)
Proof. Using the multiplication defined in Theorem 2.1, we have

\[
(\tilde{A} \otimes \tilde{x})_i = \sum_{j=1}^{n} (\tilde{a}_{ij} \otimes \tilde{x}_j)
\]

\[
= \sum_{j=1}^{n} (a_{ij}, b_{ij}, c_{ij}) \otimes (x_j, y_j, z_j)
\]

\[
= \sum_{j:a_{ij} \geq 0} (a_{ij}x_j, b_{ij}y_j, c_{ij}z_j) + \sum_{j:a_{ij} < 0, c_{ij} \geq 0} (a_{ij}z_j, b_{ij}y_j, c_{ij}x_j)
\]

\[
+ \sum_{j:c_{ij} < 0} (a_{ij}x_j, b_{ij}y_j, c_{ij}x_j)
\]

\[
= \left( \sum_{j:a_{ij} \geq 0} a_{ij}x_j + \sum_{j:a_{ij} < 0} a_{ij}z_j, \sum_{j=1}^{n} b_{ij}y_j, \sum_{j:c_{ij} \geq 0} c_{ij}z_j + \sum_{j:c_{ij} < 0} c_{ij}x_j \right)
\]

\[
\square
\]

Proposition 3.1. Suppose that the matrices \( B \) and \( M = \begin{pmatrix} A^+ & A^- \\ C^- & C^+ \end{pmatrix} \) are invertiable, and \((\tilde{x}_1, \ldots, \tilde{x}_n)^T\) given by \( \tilde{x}_j = (x_j, y_j, z_j), j = 1, \ldots, n, \) be the solution of (3.5). Then this solution is a nonnegative fuzzy exact solution of (3.4) if it satisfies \( 0 \leq x_i \leq y_i \leq z_i, i = 1, \ldots, n. \)

Proof. Using Theorem 3.1, the proof is clear. \( \square \)

According to Theorem 3.1, we know that the system (3.4), \( \tilde{A} \otimes \tilde{x} = \tilde{b}, \) is equivalent to

\[
(A^+x + A^-z, By, C^-x + C^+z) = \tilde{b} = (d_1, d_2, d_3).
\]

(3.6)

Now, suppose that one of the matrices \( B \) and \( M \) (or both) be singular or the solution of (3.6) doesn’t hold in the condition \( 0 \leq x_i \leq y_i \leq z_i, i = 1, \ldots, n. \) In this case, we may try to find approximate solution for (3.4). For this purpose, we consider the following constrained least squares problem:

\[
\min \sum_{i=1}^{n} (R((A^+x + A^-z)_i, \sum_{j=1}^{n} b_{ij}y_j, (C^-x + C^+z)_i) - R(\tilde{b}_i))^2
\]

s.t. \( \sum_{j=1}^{n} b_{ij}y_j - \sum_{j:a_{ij} \geq 0} a_{ij}x_j - \sum_{j:a_{ij} < 0} a_{ij}z_j \geq 0, \)

\( j = 1, \ldots, n, \)

\( \sum_{j,c_{ij} \geq 0} c_{ij}z_j + \sum_{j,c_{ij} < 0} c_{ij}x_j - \sum_{j=1}^{n} b_{ij}y_j \geq 0, \)

\( j = 1, \ldots, n, \)

\( \sum_{j=1}^{n} b_{ij}y_j = d_{i2}, \)

\( x_i \geq 0, \)

\( y_i - x_i \geq 0, \)

\( z_i - y_i \geq 0, \)

\( i = 1, \ldots, n. \)

(3.7)

Remark 3.1. Let for all \( i \) and \( j, 0 \in \text{supp}(\tilde{a}_{ij}), \) i.e., for all \( i \) and \( j, a_{ij} < 0 \) and \( c_{ij} \geq 0. \) Using Theorem (3.1), the system (3.4) is transformed to the following form:

\[
\begin{pmatrix}
\sum_{j=1}^{n} a_{ij}z_j \\
\sum_{j=1}^{n} b_{ij}y_j \\
\sum_{j=1}^{n} c_{ij}z_j
\end{pmatrix} = (d_{i1}, d_{i2}, d_{i3}), \quad i = 1, \ldots, n.
\]

(3.8)
Also, we can rewrite (3.8) in the matrix form as:

\[(Az, By, Cz) = (d_1, d_2, d_3).\]  

(3.9)

If \(A, B\) and \(C\) are non-singular matrices, then we obtain:

\[(z, y, z) = (A^{-1}d_1, B^{-1}d_2, C^{-1}d_3).\]  

(3.10)

It is clear that, the vector \(y\) is unique, but there are two vectors for \(z\), i.e., \(z_1 = A^{-1}d_1\) and \(z_2 = C^{-1}d_3\). In order to find the fuzzy number vector solution, we have the following cases:

(I) Let one of the two vector \(z_1\) or \(z_2\) satisfy in the conditions \(0 \leq y \leq z_1\) or \(0 \leq y \leq z_2\), respectively. For example without the loss generality, suppose \(z_1\) satisfies in the condition \(0 \leq y \leq z_1\), so we set \(x = y\), and therefore we obtain \((y, y, z_1)\) as a positive fuzzy number vector solution of FFLSE.

(II) Let both of the two vector \(z_1\) and \(z_2\) satisfy in the conditions \(0 \leq y \leq z_1\) and \(0 \leq y \leq z_2\), respectively. So we set \(x = y\) and therefore FFLSE hasn’t unique solution and two positive fuzzy number vector 
\[\tilde{S}_1 = (y, y, z_1)\]  
and 
\[\tilde{S}_2 = (y, y, z_2)\]  
are its solution. For a comparison between \(\tilde{S}_1\) and \(\tilde{S}_2\), we use a distance function to measure the closeness of the vectors \(\tilde{S}_1\) and \(\tilde{S}_2\) to \(\tilde{b}\). We will use the Ming et al. [19] metric proposed for triangular fuzzy numbers. If \(\tilde{x} = (x, \alpha_x, \beta_x)\) and \(\tilde{y} = (y, \alpha_y, \beta_y)\) are two triangular fuzzy numbers, then Ming et al. [19] introduced the distance function,

\[D_2^3(\tilde{x}, \tilde{y}) = \frac{1}{2} (4(x - y)^2 + (\alpha_y - \alpha_x)^2 + (\beta_y - \beta_x)^2) + (x - y)(\beta_y + \alpha_y - \beta_x - \alpha_x),\]  

(3.11)

and for two LR fuzzy vectors \(\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)\) and \(\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_n)\) defined the distance between \(\tilde{x}\) and \(\tilde{y}\) to be,

\[D_n^2(\tilde{x}, \tilde{y}) = \sum_{i=1}^{n} D_2^3(\tilde{x}_i, \tilde{y}_i).\]  

(3.12)

Using (3.12), we obtain \(D_n(\tilde{S}_1, \tilde{b})\) and \(D_n(\tilde{S}_2, \tilde{b})\) and choose the nearest solution to \(\tilde{b}\).

(III) Let vectors \(y\) or \(z_1\) and \(z_2\) don’t hold in condition \(y \geq 0\) or \(y \leq z_1\) and \(y \leq z_2\), respectively. In this case, we solve the constrained least squares problem (3.7) to obtain the approximation of positive fuzzy number vector solution of (3.3).
4 Numerical examples

Example 4.1. Consider the following system:

\[
\begin{align*}
(1, 2, 5) \otimes (x_1, y_1, z_1) & \otimes (3, 4, 4) \otimes (x_2, y_2, z_2) \oplus (0, 1, 2) \otimes (x_3, y_3, z_3) = (19, 68, 115) \\
(2, 3, 5) \otimes (x_1, y_1, z_1) & \otimes (0, 1, 11) \otimes (x_2, y_2, z_2) \oplus (4, 5, 6) \otimes (x_3, y_3, z_3) = (30, 77, 261) \\
(2, 5, 7) \otimes (x_1, y_1, z_1) & \otimes (4, 6, 6) \otimes (x_2, y_2, z_2) \oplus (5, 7, 10) \otimes (x_3, y_3, z_3) = (61, 167, 253)
\end{align*}
\]

So, we must solve two following systems:

\[
\begin{pmatrix}
1 & 3 & 0 & 0 & 0 & 0 \\
2 & 0 & 4 & 0 & 0 & 0 \\
2 & 4 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & 4 & 2 \\
0 & 0 & 0 & 5 & 11 & 6 \\
0 & 0 & 0 & 7 & 6 & 10
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
z_1 \\
z_2 \\
z_3
\end{pmatrix} =
\begin{pmatrix}
19 \\
2945 \\
61 \\
115 \\
261 \\
253
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
2 & 4 & 1 \\
3 & 1 & 5 \\
5 & 6 & 7
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix} =
\begin{pmatrix}
68 \\
77 \\
167
\end{pmatrix}
\]

Using (3.6), we have

\[
\begin{pmatrix}
(x_1, y_1, z_1) \\
(x_2, y_2, z_2) \\
(x_3, y_3, z_3)
\end{pmatrix} =
\begin{pmatrix}
(1, 2, 7) \\
(6, 12, 14) \\
(7, 10, 12)
\end{pmatrix}
\]

as a positive fuzzy number vector solution of (4.13).

Example 4.2. Consider the following system:

\[
\begin{align*}
(5.6621, 6.2945, 7.2520) \otimes (x_1, y_1, z_1) & \oplus (-7.7388, -7.4603, -7.3027) \otimes (x_2, y_2, z_2) = (6.5421, 24.0383, 35.5858) \\
(8.0183, 8.1185, 9.0807) \otimes (x_1, y_1, z_1) & \oplus (7.7206, 8.2675, 8.2675) \otimes (x_2, y_2, z_2) = (100.8751, 117.8106, 134.2153)
\end{align*}
\]

So, we must solve two following systems:

\[
\begin{pmatrix}
5.6621 & 0 & 8.0183 & 7.7206 \\
0 & -7.7388 & 0 & 0 \\
0 & -7.3027 & 0 & 0 \\
7.2520 & 0 & 9.0807 & 8.2675
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
z_1 \\
z_2
\end{pmatrix} =
\begin{pmatrix}
6.5421 \\
100.8751 \\
35.5858 \\
134.2153
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
6.2945 & -7.4603 \\
8.1185 & 8.2675
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix} =
\begin{pmatrix}
24.0383 \\
117.8106
\end{pmatrix}
\]
Using (3.7), we have
\[
\begin{bmatrix}
(x_1, y_1, z_1) \\
(x_2, y_2, z_2)
\end{bmatrix} = \begin{bmatrix}
0.1002069, 9.569978, 16.56075 \\
0, 4.852342, 10.14861
\end{bmatrix}
\]
as a positive fuzzy number vector solution of (4.14) but the proposed method by kumar [17] has not solution. Also, the objective value of (3.7) is \(0.6812963 \times 10^{-20}\).

Example 4.3. Consider the following system:
\[
\begin{align*}
(-10, -7, -4) \odot (x_1, y_1, z_1) \odot (-7, -5, -3) \odot (x_2, y_2, z_2) & \odot (-5, -2, -1) \odot (x_3, y_3, z_3) \\
& = (-36, -36, -33)
\\
(-6, -4, -2) \odot (x_1, y_1, z_1) \odot (-4, -3, -1) \odot (x_2, y_2, z_2) & \odot (-8, -5, -5) \odot (x_3, y_3, z_3) \\
& = (-26, -25, -23)
\\
(-8, -7, -3) \odot (x_1, y_1, z_1) \odot (-11, -1, -1) \odot (x_2, y_2, z_2) & \odot (-3, -2, -1) \odot (x_3, y_3, z_3) \\
& = (-44, -21, -19)
\end{align*}
\]
(4.15)

Using (3.6), we have
\[
\begin{bmatrix}
(x_1, y_1, z_1) \\
(x_2, y_2, z_2) \\
(x_3, y_3, z_3)
\end{bmatrix} = \begin{bmatrix}
(1, 2, 4) \\
(3, 4, 5) \\
(1, 1, 2)
\end{bmatrix}
\]
as a positive fuzzy number vector solution of (4.15).

Example 4.4. Consider the following system:
\[
\begin{align*}
(4.9950, 5.8441, 6.5873) \odot (x_1, y_1, z_1) & \odot (2.4361, 3.1148, 3.7703) \odot (x_2, y_2, z_2) \\
& = (34.0822, 42.2359, 48.4583)
\\
(8.2559, 9.1898, 9.5821) \odot (x_1, y_1, z_1) & \odot (-10.1400, -9.2500, -9.1180) \odot (x_2, y_2, z_2) \\
& = (50.2106, 61.9286, 64.9654)
\end{align*}
\]
(4.16)

Using (3.7), we have
\[
\begin{bmatrix}
(x_1, y_1, z_1) \\
(x_2, y_2, z_2)
\end{bmatrix} = \begin{bmatrix}
0.2578402, 7.058067, 12.14679 \\
0.5868982 \times 10^{-2}, 0.3171481, 0.3245293
\end{bmatrix}
\]
as a positive fuzzy number vector solution of (4.16) but the proposed method by kumar [17] has not solution. Also, the objective value is of (3.7) \(0.4089455 \times 10^{-26}\).

Example 4.5. Consider the following system:
\[
\begin{align*}
(-2, 3, 4) \odot (x_1, y_1, z_1) & \odot (-3, -2, -1) \odot (x_2, y_2, z_2) = (5, 21, 43)
\\
(-1, 1, 2) \odot (x_1, y_1, z_1) & \odot (1, 3, 4) \odot (x_2, y_2, z_2) = (-6, 14, 34)
\end{align*}
\]
(4.17)
Using (3.7), we have

\[
\begin{bmatrix}
(x_1, y_1, z_1) \\
(x_2, y_2, z_2)
\end{bmatrix} = \begin{bmatrix}
(0, 8.272727, 46.22885) \\
(0, 1.909091, 14.81923)
\end{bmatrix}
\]

as a positive fuzzy number vector solution of (4.17) but the proposed method by kumar [17] has not solution.

Example 4.6. Consider the following system:

\[
\begin{align*}
(-1,0,3) \odot (x_1, y_1, z_1) \odot (-3,1,2) \odot (x_2, y_2, z_2) & \quad \odot (-2,0,0) \odot (x_3, y_3, z_3) \\
(-2,-1,2) \odot (x_1, y_1, z_1) \odot (-1,2,5) \odot (x_2, y_2, z_2) & \quad \odot (-5,3,4) \odot (x_3, y_3, z_3) \\
(-4,1,2) \odot (x_1, y_1, z_1) \odot (-2,5,7) \odot (x_2, y_2, z_2) & \quad \odot (-1,2,3) \odot (x_3, y_3, z_3)
\end{align*}
\]

\[
= \begin{bmatrix}
(-31,6,24) \\
(-43,38,75) \\
(-32,59,69)
\end{bmatrix}
\]

Using (3.6), we have

\[
\begin{bmatrix}
(x_1, y_1, z_1) \\
(x_2, y_2, z_2) \\
(x_3, y_3, z_3)
\end{bmatrix} = \begin{bmatrix}
(7,7,8) \\
(6,6,10) \\
(11,11,12)
\end{bmatrix}
\]

as a positive fuzzy number vector solution of (4.18).

5 Conclusion

In this paper, we used arithmetic operations on fuzzy numbers that introduced by Kaffman [18] and found the positive fuzzy number vector solution for the fully fuzzy linear system of equations. First, we converted the FFLSE into two crisp systems and if both coefficient matrices be non-singular then we solved the crisp systems. By allocating the vector solution of $2n \times 2n$ system to the vector solution of $n \times n$ system, we obtained the vector solution of the FFLSE. If the vector solutions were acceptable (Solution be positive and satisfies the fuzzy number condition), we set that the fuzzy numbers vector solution of FFLSE. Else we solved the constrained least squares problem to found the optimal fuzzy numbers vector solution. In comparison with the proposed method by kumar [17], our proposed method is very useful and the FFLSE always has positive fuzzy numbers vector solution and we showed this capability by solving some numerical examples.

References

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