Fuzzy Differential Equations with Generalized Derivative

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Abstract

In this paper the new concept of generalized derivative for fuzzy mappings is entered and fuzzy differential equations with generalized derivative are considered. The existence theorem is proved. Some model examples are considered to explain the obtained results.

Keywords: Generalized derivative; fuzzy equation; existence theorem

1 Introduction

The concept of fuzzy derivative was first entered by S.S.L. Chang and L. Zadeh [4]. Then the problems of differentiability of fuzzy mappings were considered by D. Dubois and H. Prade [6], O. Kaleva [9], R. Goetschel and W. Voxman [7], S. Seikkala [20], A. Kandel, M. Friedman and M. Ming [10], B. Bede and S.G. Gal [2], M.L. Puri and D.A. Ralescu [16], J.J. Buckley and Th. Feuring [3] etc. However earlier considered fuzzy differential equations used fuzzy Hukuhara derivative [9, 12, 13, 15] and generalized derivative [1, 2, 5, 19].

In section 2 the new concept of generalized derivative for fuzzy mappings which is based on a derivative for set-valued mappings [14] is entered. In section 3 the fuzzy differential equations with generalized derivative are considered. The existence theorem is proved. Model examples are considered to explain the obtained results.
2 Preliminaries

Let $CC(\mathbb{R}^n)$ be the family of all nonempty compact convex subsets of $\mathbb{R}^n$ with the Hausdorff metric

$$h(A, B) = \max\{\max_{a \in A} \min_{b \in B} \|a - b\|, \max_{b \in B} \min_{a \in A} \|a - b\|\},$$

where $\| \cdot \|$ denotes the usual Euclidean norm in $\mathbb{R}^n$.

Let $\mathbb{E}^n$ be the family of mappings $x : \mathbb{R}^n \to [0, 1]$ satisfying the following conditions:

(i) $x$ is normal, i.e. there exists an $\xi_0 \in \mathbb{R}^n$ such that $x(\xi_0) = 1$;

(ii) $x$ is fuzzy convex, i.e. $x(\lambda \xi + (1 - \lambda) \zeta) \geq \min\{x(\xi), x(\zeta)\}$ whenever $\xi, \zeta \in \mathbb{R}^n$ and $\lambda \in [0, 1]$;

(iii) $x$ is upper semicontinuous, i.e. for any $\xi_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ exists $\delta(\xi_0, \varepsilon) > 0$ such that $x(\zeta) < x(\xi_0) + \varepsilon$ whenever $||\zeta - \xi_0|| < \delta$, $\xi \in \mathbb{R}^n$;

(iv) the closure of the set $\{\xi \in \mathbb{R}^n : x(\xi) > 1\}$ is compact.

Let $\hat{0}$ be the fuzzy mapping defined by $\hat{0}(\xi) = 0$ if $\xi \neq 0$ and $\hat{0}(0) = 1$.

**Definition 2.1.** The set $\{y \in \mathbb{R}^n : x(y) \geq \alpha\}$ is called the $\alpha$-level $[x]^\alpha$ of a mapping $x \in \mathbb{E}^n$ for $0 < \alpha \leq 1$. The closure of the set $\{y \in \mathbb{R}^n : x(y) > 0\}$ is called the $0$-level $[x]_0^0$ of a mapping $x \in \mathbb{E}^n$.

Define the metric $D : \mathbb{E}^n \times \mathbb{E}^n \to \mathbb{R}_+$ by the equation $D(x, y) = \sup_{\alpha \in [0, 1]} h([x]^\alpha, [y]^\alpha)$.

Using the results of [17], we know that

(i) $(\mathbb{E}^n, D)$ is a complete metric space,

(ii) $D(x + z, y + z) = D(x, y)$ for all $x, y, z \in \mathbb{E}^n$,

(iii) $D(kx, ky) = |k|D(x, y)$ for all $x, y \in \mathbb{E}^n$, $k \in \mathbb{R}$.

Let $A, B, C$ be in $CC(\mathbb{R}^n)$. The set $C$ is called the Hukuhara difference of $A$ and $B$, if $B + C = A$, i.e. $C = A - B$ [8]. From Rådström’s Cancellation Lemma [18], it follows that if this difference exists, then it is unique.

Let $X : [0, T] \to CC(\mathbb{R}^n)$ be a set-valued mapping; $(t_0 - \Delta, t_0 + \Delta) \subset [0, T]$ be a $\Delta$-neighborhood of a point $t_0 \in [0, T]$; $\Delta > 0$.

For any $t \in (t_0 - \Delta, t_0 + \Delta)$ consider the following Hukuhara differences

$$X(t) \xrightarrow{h} X(t_0), \ t \geq t_0, \tag{2.1}$$

$$X(t_0) \xrightarrow{h} X(t), \ t \geq t_0, \tag{2.2}$$

$$X(t_0) \xrightarrow{h} X(t), \ t \leq t_0, \tag{2.3}$$

$$X(t) \xrightarrow{h} X(t_0), \ t \leq t_0 \tag{2.4}$$

if these differences exist.
Definition 2.2. The differences (2.1) and (2.2) ((2.3) and (2.4)) are called the right (left) differences.

Remark 2.1. From the properties of the Hukuhara difference it follows that both one-sided differences exist only in the case when \( X(t) = A + \{ f(t) \} \) for \( t \in [t_0, t_0 + \Delta) \) or \( t \in (t_0 - \Delta, t_0] \), where \( A \in \text{CC}(\mathbb{R}^n) \) and \( f : [0, T] \to \mathbb{R}^n \). If all differences (2.1) - (2.4) exist then \( X(t) = A + \{ f(t) \} \) in \( \Delta \) - neighborhood of the point \( t_0 \).

If for all \( t \in (t_0 - \Delta, t_0 + \Delta) \) there exists only one of the one-sided differences, then using the properties of the Hukuhara difference, we get that the mapping \( \text{diam}(X) : [0, T] \to \mathbb{R}_+ \) in the \( \Delta \) - neighborhood of the point \( t_0 \) can be:

a) non-decreasing on \( (t_0 - \Delta, t_0 + \Delta) \);

b) non-increasing on \( (t_0 - \Delta, t_0 + \Delta) \);

c) non-decreasing on \( (t_0 - \Delta, t_0) \) and non-increasing on \( (t_0, t_0 + \Delta) \);

d) non-increasing on \( (t_0 - \Delta, t_0) \) and non-decreasing on \( (t_0, t_0 + \Delta) \).

Hence, for each of the above mentioned cases only one of combinations of differences is possible:

a) (2.1) and (2.3);

b) (2.2) and (2.4);

c) (2.2) and (2.3);

d) (2.1) and (2.4).

Proposition 2.1. In a case when both of the one-sided differences exist we will choose those one that corresponds to a combination having a smaller serial number.

Accordingly there are four types of limits

\[
\lim_{t \to t_0} \frac{1}{t - t_0} (X(t) - X(t_0)), \quad (2.5)
\]

\[
\lim_{t \to t_0} \frac{1}{t - t_0} (X(t_0) - X(t)), \quad (2.6)
\]

\[
\lim_{t \to t_0} \frac{1}{t_0 - t} (X(t) - X(t)), \quad (2.7)
\]

\[
\lim_{t \to t_0} \frac{1}{t_0 - t} (X(t) - X(t_0)), \quad (2.8)
\]

Taking into account Proposition 2.1, it is possible to say that in the point \( t_0 \) not more than two limits can exist (as we assumed that there exist only two of four Hukuhara differences).

Remark 2.2. Considering all above we have that there exist only the following possible combinations: a) (2.5) and (2.7); b) (2.6) and (2.8); c) (2.6) and (2.7); d) (2.5) and (2.8).

Definition 2.3. [14], If the corresponding two limits exist and are equal we will say that the mapping \( X(\cdot) \) is differentiable in the generalized sense in the point \( t_0 \) and denote the generalized derivative by \( DX(t_0) \).

Definition 2.4. [14], The set-valued mapping \( X : [0, T] \to \text{CC}(\mathbb{R}^n) \) is said to be differentiable in the generalized sense on the interval \([0, T] \) if it is differentiable in the generalized sense at every point of this interval.
Consider the fuzzy mapping $x : [0, T] \mapsto E^n$ if for any $\alpha \in [0, 1]$ the set-valued mapping $x_\alpha(t) = [x(t)]^\alpha$ is differentiable in the generalized sense at point $t$ with $Dx_\alpha(t)$ and the family $\{Dx_\alpha : \alpha \in [0, 1]\}$ define a fuzzy number $\hat{x}(t) \in E^n$.

If $x : [0, T] \mapsto E^n$ is generalized differentiable at $t \in [0, T]$, then we say that $\dot{x}(t)$ is the fuzzy derivative of $x(\cdot)$ at the point $t \in [0, T]$.

**Corollary 2.1.** If the fuzzy mapping $x(\cdot)$ is differentiable in the generalized sense on $[0, T]$ and $\text{diam}([x(\cdot)]^\alpha)$ is a non-decreasing function on $[0, T]$ for all $\alpha \in [0, 1]$ then the fuzzy mapping $x(\cdot)$ is Hukuhara fuzzy differentiable [9] and $\dot{x}(t) = D_Hx(t)$.

**Proof.** As the fuzzy mapping $x : [0, T] \mapsto E^n$ is differentiable in the generalized sense and $\text{diam}([x(\cdot)]^\alpha)$ is a non-decreasing function on $[0, T]$ for all $\alpha \in [0, 1]$ limits (2.5) and (2.7) exist and are equal.

Hence, the mapping $x(\cdot)$ is Hukuhara fuzzy differentiable by the definition. \hfill \square

**Corollary 2.2.** If the fuzzy mapping $x(\cdot)$ is Hukuhara fuzzy differentiable it is differentiable in the generalized sense and $\dot{x}(t) = D_Hx(t)$.

**Proof.** From the definition of the Hukuhara fuzzy derivative the existence and the equality of limits (2.5) and (2.7) follows. Hence, by the definition the mapping $x(\cdot)$ is differentiable in the generalized sense. \hfill \square

**Remark 2.3.** There exist fuzzy mappings that are differentiable in the generalized sense but not Hukuhara fuzzy differentiable.

**Example 2.1.** Consider the fuzzy mapping $x(t)$ such that for all $\alpha \in [0, 1]$

$$[x(t)]^\alpha = S(1-\alpha)\sin t(0), \quad t \in (0, \pi),$$

where $S_r(a) = \{ \xi \in \mathbb{R}^n \mid \|\xi - a\| \leq r \}$. It is obvious that the given fuzzy mapping is Hukuhara fuzzy differentiable only on the interval $[0, \pi/2]$ and $[D_Hx(t)]^\alpha = S(1-\alpha)\cos t(0)$ for all $\alpha \in [0, 1]$. On the interval $(\pi/2, \pi)$ it is not Hukuhara fuzzy differentiable as its diameter on this interval decreases.

However, using the definition of the generalized differentiability, we will receive $\dot{x}(t)$ such that

$$[\dot{x}(t)]^\alpha = \begin{cases} S(1-\alpha)\cos t(0), & t \in (0, \pi/2), \\ \{0\}, & t = \pi/2, \\ S(1-\alpha)\cos t(0), & t \in (\pi/2, \pi) \end{cases}$$

**Example 2.2.** The fuzzy mapping $x(t)$ such that $[x(t)]^\alpha = S(1-\alpha)\delta(0)$ is differentiable in the generalized sense on $\mathbb{R}^1$ and has generalized derivative $\dot{x}(t)$ such that $[\dot{x}(t)]^\alpha = S_{1-\alpha}(0)$.

### 3 The fuzzy differential equation

First consider a fuzzy differential equation with the generalized derivative that is similar to a fuzzy differential equation with the Hukuhara fuzzy derivative, i.e.

$$\dot{x}(t) = f(t, x(t)), \quad x(0) = x_0,$$

(3.9)
where \( \dot{x}(t) \) is the generalized derivative of a fuzzy mapping \( x : [0, T] \rightarrow \mathbb{E}^n, f : [0, T] \times \mathbb{E}^n \rightarrow \mathbb{E}^n \) is a fuzzy mapping, \( x_0 \in \mathbb{E}^n \).

**Definition 3.1.** A fuzzy mapping \( x : [0, T] \rightarrow \mathbb{E}^n \) is said to be solution of fuzzy differential equation (3.9) if it is absolutely continuous and satisfies (3.9) almost everywhere on \([0, T]\).

**Remark 3.1.** Unlike the case of fuzzy differential equations with Hukuhara fuzzy derivative, if a fuzzy differential equation with the generalized derivative (3.9) has a solution then there exists an infinite number of solutions irrespective of the conditions on the right-hand side of the equation.

**Example 3.1.** Consider the following fuzzy differential equation with the generalized derivative

\[
\dot{x}(t) = a, \ x(0) = b, \tag{3.10}
\]

where \( a, b \in \mathbb{E}^1 \) such that \([a]^{\alpha} = [-1 + \alpha, 1 - \alpha], [b]^{\alpha} = [-2 + 2\alpha, 2 - 2\alpha] \) for all \( \alpha \in [0, 1] \).

It is easy to check that the following fuzzy mappings are the solutions of equation (3.10):

\[
x_1(t) \text{ such that } [x_1(t)]^{\alpha} = [(-2 + 2\alpha) - (1 - \alpha)t, (2 - 2\alpha) + (1 - \alpha)t], \ t \in [0, 1],
\]

\[
x_2(t) \text{ such that } [x_2(t)]^{\alpha} = [(-2 + 2\alpha) + (1 - \alpha)t, (2 - 2\alpha) - (1 - \alpha)t], \ t \in [0, 1],
\]

\[
x_3(t) \text{ such that } [x_3(t)]^{\alpha} = \begin{cases} 
[(-2 + 2\alpha) - (1 - \alpha)t, (2 - 2\alpha) + (1 - \alpha)t], & t \in [0, 0.5], \\
[(-2.5 + 2.5\alpha) + (1 - \alpha)t, (2.5 - 2.5\alpha) - (1 - \alpha)t], & t \in (0.5, 1],
\end{cases}
\]

and so on.

Also it is possible to construct other solutions, thus only \( x_1(\cdot) \) will be the solution of the corresponding fuzzy differential equation with the Hukuhara fuzzy derivative

\[ D_H x(t) = a, \ x(0) = b. \]

Therefore we will consider the other fuzzy differential equation with the generalized derivative

\[
\dot{x}(t) \overset{\text{h}}{=} \Phi(\varphi(t))f(t, x(t)) = \Phi(-\varphi(t))f(t, x(t)), \ x(0) = x_0, \tag{3.11}
\]

where \( t \in [0, T]; f : [0, T] \times \mathbb{E}^n \rightarrow \mathbb{E}^n \) is the fuzzy mapping, \( x_0 \in \mathbb{E}^n; \varphi : [0, T] \rightarrow \mathbb{R}^1 \) is a continuous function, function \( \Phi : \mathbb{R}^1 \rightarrow \{0, 1\} \) such that

\[
\Phi(\varphi) = \begin{cases} 
1, & \varphi > 0, \\
0, & \varphi \leq 0.
\end{cases}
\]

**Remark 3.2.** It is obvious that the fuzzy mapping \( f(t, x) \) defines only on "how much" the fuzzy mapping \( x(\cdot) \) changes in case of its "decrease" or "increase" and function \( \varphi(\cdot) \) defines what will be with \( x(\cdot) \) ["decrease" or "increase"]. If \( \varphi(t) \equiv 0 \) irrespective of \( f(t, x(t)) \) the fuzzy mapping \( x(\cdot) \) will be constant.

**Definition 3.2.** A fuzzy mapping \( x : [0, T] \rightarrow \mathbb{E}^n \) is called the solution of differential equation (3.11) if it is absolutely continuous, satisfies (3.11) almost everywhere on \([0, T]\).
and
\[
\text{diam}[x(t)]^\alpha = \begin{cases} 
\text{increases} & \text{if } \varphi(t) > 0, \\
\text{constant} & \text{if } \varphi(t) = 0, \\
\text{decreases} & \text{if } \varphi(t) < 0
\end{cases}
\]
for any \( \alpha \in [0, 1] \).

If on the interval \([\tau_1, \tau_2]\) the function \(\varphi(t) > 0\) and \(\text{diam}[x(t)]^\alpha\) increases for any \(\alpha \in [0, 1]\), then we have
\[
x(t) = x(\tau_1) + \int_{\tau_1}^{t} f(s, x(s))ds = \hat{0}
\]
for \(t \in [\tau_1, \tau_2]\), i.e.
\[
x(t) = x(\tau_1) + \int_{\tau_1}^{t} f(s, x(s))ds.
\]

If on the interval \([\tau_1, \tau_2]\) the function \(\varphi(t) < 0\) and \(\text{diam}[x(t)]^\alpha\) decreases for any \(\alpha \in [0, 1]\), then we have
\[
x(t) = x(\tau_1) - \int_{\tau_1}^{t} f(s, x(s))ds,
\]
i.e.
\[
x(t) = x(\tau_1) - \int_{\tau_1}^{t} f(s, x(s))ds.
\]

If on the interval \([\tau_1, \tau_2]\) the function \(\varphi(t) \equiv 0\), then we have \(x(t) = x(\tau_1)\) for all \(t \in [\tau_1, \tau_2]\). Hence, we can make other equivalent definition of a solution of the system (3.11).

**Definition 3.3.** A fuzzy mapping \(x : [0, T] \rightarrow \mathbb{E}^n\) is a solution to the problem (3.11) if and only if it is continuous and on any subinterval \([\tau_1, \tau_2] \subset [0, T]\), where function \(\varphi(t)\) has a constant sign, satisfies the integral equation
\[
x(t) + \int_{\tau_1}^{t} \Phi(-\varphi(s))f(s, x(s))ds = x(\tau_1) + \int_{\tau_1}^{t} \Phi(\varphi(s))f(s, x(s))ds
\]
for all \(t \in [\tau_1, \tau_2]\).

**Example 3.2.** Consider the following fuzzy differential equation with generalized derivative
\[
\dot{x}^h \Phi(\sin(t))a = \Phi(-\sin(t))a, \quad x(0) = b, \quad (3.12)
\]
where \(x \in \mathbb{E}^1, t \in [0, 2\pi], a, b \in \mathbb{E}^1\) such that
\[
[a]^\alpha = [1 + \alpha, 3 - \alpha], \quad [b]^\alpha = [2 + \alpha, 4 - \alpha].
\]
for all $\alpha \in [0, 1]$. As $\sin(t) > 0$ for $t \in (0, \pi)$ we have

$$x(t) = b + \int_0^t ads$$

for $t \in [0, \pi]$, i.e.

$$x(t) : [x(t)]^\alpha = [2 + \alpha + (1 + \alpha)t, 4 - \alpha + (3 - \alpha)t]$$

for $t \in [0, \pi]$ and $\alpha \in [0, 1]$.

Further as $\sin(t) < 0$ for $t \in (\pi, 2\pi)$ we have

$$x(t) + \int_\pi^t ads = c,$$

where $c : [c]^\alpha = [2 + \pi + \alpha + \alpha\pi, 4 + 3\pi - \alpha - \alpha\pi]$ for $\alpha \in [0, 1]$.

Therefore, $x(t) = c - \frac{h}{\alpha} \int_\pi^t ads$ for $t \in [\pi, 2\pi]$, i.e.

$$x(t) : [x(t)]^\alpha = [2 + 2\pi + \alpha + 2\alpha\pi - t - \alpha t, 4 + 6\pi - \alpha - 2\alpha\pi - 3t + \alpha t]$$

for $t \in [\pi, 2\pi]$ and $\alpha \in [0, 1]$.

So for $t = 2\pi$ we get $x(2\pi) = b$. If we consider this equation for $t \in [0, +\infty)$ we will get the periodic solution.

**Example 3.3.** Consider the same fuzzy differential equation with generalized derivative but with $\varphi(t) = \cos(t)$:

$$\dot{x} = \Phi(\cos(t))a = \Phi(-\cos(t))a, \ x(0) = b. \quad (3.13)$$

As $\cos(t) > 0$ for $t \in (0, \frac{\pi}{2})$ then we have

$$x(t) = b + \int_0^t ads$$

for $t \in [0, \frac{\pi}{2}]$, i.e.

$$x(t) : [x(t)]^\alpha = [2 + \alpha + (1 + \alpha)t, 4 - \alpha + (3 - \alpha)t]$$

for $t \in [0, \frac{\pi}{2}]$ and $\alpha \in [0, 1]$.

Further as $\cos(t) < 0$ for $t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ then we get

$$x(t) + \int_\frac{\pi}{2}^t ads = c,$$
Consider the differential equation from Example 3.2 with
\[ \Phi(t) = \begin{cases} 
0 & \text{for } t \in [0, 2\pi] \\
1 & \text{for } t \in (2\pi, 4\pi] \\
2 & \text{otherwise} 
\end{cases} \]
for all \( t \in [\frac{\pi}{2}, 1 + \pi] \) and \( \alpha \in [0, 1] \).
So for \( t = 1 + \pi < \frac{3\pi}{2} \) we have \( x(1 + \pi) = 2 \), where \( d \in \mathbb{E}^1 : [d]^{\alpha} = \{1\} \) for \( \alpha \in [0, 1] \). It means that the fuzzy solution exists only for \( t \in [0, 1 + \pi] \).

**Example 3.4.** Consider the differential equation from Example 3.2 with \( \varphi(t) \equiv 0 \) for \( t \in [0, 2\pi] \). Then \( \Phi(\varphi(t)) = 0 \) and we have \( x(t) = b \) for \( t \in [0, 2\pi] \).

**Example 3.5.** Consider the differential equation from Example 3.2 with \( \varphi(t) \equiv -1 \). Then we have
\[ x(t) + \int_{0}^{t} \alpha \cdot ds = b. \tag{3.14} \]
Therefore \( x(t) \) such that \( |x(t)|^{\alpha} = [2 + \alpha - t - \alpha t, 4 - \alpha - 3\alpha + \alpha t] \) for all \( \alpha \in [0, 1] \).
Then for \( t = 1 \) we get \( x(1) = d \), where \( d \in \mathbb{E}^1 : [d]^{\alpha} = \{1\} \) for \( \alpha \in [0, 1] \). So the solution exists for \( t \in [0, 1] \).

**Remark 3.3.** Obviously, for all \( \varphi(t) \) we can guarantee the existence of solution of the fuzzy differential equation \( \dot{x} + \Phi(\varphi(t)) \cdot a = \Phi(-\varphi(t)) \cdot a, x(0) = b \) on interval \([0, 1]\).

The following theorem of existence of the solution of equation (3.11) holds:

**Theorem 3.1.** Let the fuzzy mapping \( f(t, x) \) defined in the domain \( Q = \{(t, x) \in \mathbb{R}^1 \times \mathbb{E}^n : 0 \leq t \leq r, D(x, x_0) \leq p\} \) satisfy the following conditions:

\( a) \) for any fixed \( x \in Q \) the fuzzy mapping \( f(\cdot, x) \) is measurable;

\( b) \) for almost every fixed \( t \in [0, r] \) the fuzzy mapping \( f(t, \cdot) \) is continuous;

\( c) \) \( D(f(t, x), 0) \leq m(t) \), where \( m(\cdot) \) is summable on \([0, r]\);

\( d) \) \( \varphi : [0, r] \rightarrow \mathbb{R}^1 \) is continuous and the equation \( \varphi(t) = 0 \) has finite number of roots on the segment \([0, r]\);

\( e) \) if \( \varphi(s) < 0, s \in [0, r] \) and

1) \( \text{diam}([x]^{\alpha}) = 0 \), then \( \text{diam}([f(s, x)]^{\alpha}) = 0 \);

2) \( \text{diam}([x]^{\alpha}) \neq 0 \), then there exists \( \Delta_1 > 0 \) such that \( \varphi(\tau) \leq 0 \) for all \( \tau \in [s, s + \Delta_1] \)
and there exists the difference
\[ \frac{x}{h} \int_{\tau}^{\tau+\Delta} f(s, x) ds \]
for all \( \tau \in [s, s + \Delta_1], \Delta < \Delta_1 \) such that \( \tau + \Delta \leq s + \Delta_1 \). Then there exists a solution of equation (3.11) defined on the interval \([0, q]\), where \( q > 0 \) satisfies the conditions
\[ q \leq r, \psi(q) \leq p, \psi(t) = \int_{0}^{t} m(s) ds. \]
Proof. Consider the behavior of the function \( \varphi(t) \) on the interval \([0, q]\). As \( \varphi(t) \) is continuous the equation \( \varphi(t) = 0 \) has the finite quantity of solutions on \([0, q]\);

First suppose that \( \varphi(t) \) changes its sign on the interval \([0, q]\) once, i.e. there exists \( \tau \in (0, q) \) such that \( \varphi(\tau) = 0 \) and

1) \( \varphi(t) < 0, \ t \in [0, \tau) \) and \( \varphi(t) > 0, \ t \in (\tau, q] \);

or

2) \( \varphi(t) > 0, \ t \in [0, \tau) \) and \( \varphi(t) < 0, \ t \in (\tau, q] \).

Consider the first case. Choose any natural \( k \) such that \( \Delta = \frac{\tau}{k} < \Delta_1 \). Sequentially on the intervals \( i\Delta \leq t \leq (i+1)\Delta, \ i = 0, 1, \ldots, k-1 \), let us build the successive approximations of the solution using condition e) of the theorem:

\[
x^k(t) = x_0 \text{ for } t \leq 0,
\]

\[
x^k(t) = x_0 \frac{h}{0}^{t} f(s, x^k(s - \Delta))ds \text{ for } t_0 \leq t \leq \tau. \tag{3.15}
\]

Let us show that for any \( k \) the fuzzy mapping \( x^k(t) \) is defined, continuous and satisfies the inequality

\[
D(x^k(t), x_0) \leq \psi(\tau) \leq b \text{ for } t_0 \leq t \leq \tau. \tag{3.16}
\]

We will use the method of full mathematical induction. Let \( t \in [0, \Delta] \) then using (3.15), we have

\[
x^k(t) = x_0 \frac{h}{0}^{t} f(s, x_0)ds.
\]

As the fuzzy mapping \( f(t, x_0) \) is measurable on \([0, \Delta]\) then \( x^k(t) \) is defined and continuous.

Besides

\[
D(x^k(t), x_0) \leq \int_{0}^{t} D \left( f(s, x_0), \hat{0} \right) ds \leq \int_{0}^{t} m(s)ds \leq \psi(\tau) \leq p.
\]

Let us assume that the fuzzy mapping \( x^k(t) \) satisfies the mentioned conditions on the interval \([0, i\Delta] \). Now we choose any \( t \in [0, (i+1)\Delta] \). Then the fuzzy mapping \( f(t, x^k(t - \Delta)) \) is measurable. Hence, \( x^k(t) \) is defined and continuous. Besides, we have

\[
D(x^k(t), x_0) \leq \int_{0}^{t} D \left( f(s, x^k(s - \Delta)), \hat{0} \right) ds \leq \int_{0}^{t} m(s)ds \leq \psi(\tau) \leq p.
\]

Thus, the above statement is valid.

From (3.16) it follows that the sequence of the set-valued mappings \( x^k(t), k = 1, \ldots, \infty \) in uniformly bounded:

\[
D(x^k(t), \hat{0}) \leq D(x_0, \hat{0}) + \psi(\tau) \leq D(x_0, \hat{0}) + p.
\]

Let us show that the fuzzy mappings \( x^k(t) \) are equicontinuous. For any \( \gamma, \xi \in [0, \tau], \gamma \leq \xi \)
and any natural $k$ the inequality holds

$$D(x^k(\xi), x^k(\gamma)) = D\left(x_0 h \int_0^\xi f(s, x^k(s - \Delta)) ds, x_0 h \int_0^\gamma f(s, x^k(s - \Delta)) ds\right)$$

$$= D\left(\int_\gamma^\xi f(s, x^k(s - \Delta)) ds, \hat{0}\right)$$

$$\leq \int_\gamma^\xi D\left(f(s, x^k(s - \Delta)), \hat{0}\right) ds$$

$$\leq \int_\gamma^\xi m(s) ds$$

$$= \psi(\xi) - \psi(\gamma).$$

(3.17)

The function $\psi(t)$ is absolutely continuous on $[0, \tau]$ as the integral of the summable function with a variable top limit. Hence, for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all $\gamma, \xi \in [0, \tau] : 0 \leq \xi - \gamma < \delta$ the inequality $\psi(\xi) - \psi(\gamma) < \varepsilon$ is fair. Then using (3.17), we get $D(x^k(\xi), x^k(\gamma)) < \varepsilon$ for $0 \leq \xi - \gamma < \delta$, i.e. the sequence $\{x^k(t)\}_{k=1}^\infty$ is equicontinuous.

According to Ascoli theorem [11] we can choose a uniformly converging subsequence of the sequence $\{x^k(t)\}_{k=1}^\infty$. Its limit is a continuous fuzzy mapping that we will denote by $x(t)$. As

$$D(x^k(s - \Delta), x(s)) \leq D(x^k(s - \Delta), x^k(s)) + D(x^k(s), x(s)),$$

and the first summand is less than $\varepsilon$ for $\Delta < \delta(\varepsilon)$ in view of the equicontinuity of the fuzzy mappings $\{x^k(t)\}$, then the chosen subsequence $\{x^k(s - \Delta)\}$ converges to $x(s)$. Owing to the theorem conditions in (3.15) it is possible to pass to the limit under the sign of the integral. We obtain that the fuzzy mapping $x(t)$ satisfies equation (3.11) and $x(0) = x_0$, i.e. $x(t)$ is the solution of (3.11) on the interval $[0, \tau]$.

Now it is necessary for us to prove the existence of a solution on the interval $[\tau, q]$. It is obvious that the equation

$$\dot{x}(t) = \hat{0}(\Phi(\varphi(t))) f(t, x(t)) = \Phi(-\varphi(t)) f(t, x(t)), \ x(\tau) = x_\tau$$

on the interval $[\tau, q]$ turns to be the usual fuzzy differential equation with Hukuhara fuzzy derivative

$$\dot{x}(t) = f(t, x(t)), \ x(\tau) = x_\tau$$

and the proof follows from [15].

It is obvious that in the second case the proof is similar, but straightforward.

When the function $\varphi(t)$ changes its sign more than once on the interval $[0, q]$ the proof is similar.

If the function $\varphi(t)$ does not change its sign on the interval $[0, q]$:

in case when $\varphi(t) > 0$, (3.11) turns to the fuzzy differential equation with Hukuhara fuzzy derivative and the existence of the solution on the interval $[0, q]$ follows from [15];

in a case when $\varphi(t) < 0$ the proof of the existence of the solution on the interval $[0, q]$ is similar to the first part of the proof above. □
4 Conclusion

The considered fuzzy differential equations generalize the fuzzy differential equations with the fuzzy Hukuhara derivative (HFDE) and generalized derivative (GFDE). If function $\varphi(t) > 0$, then the solution of this equation coincides with HFDE and GFDE. If function $\varphi(t) < 0$, then the solution of this equation coincides with GFDE. If function $\varphi(t)$ changes a sign then no such solutions have HFDE and GFDE.

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