Some Duality Results for Fuzzy Nonlinear Programming Problem

G. Panda 1,*; S. Jaiswal 2

(1) Department of Mathematics, Indian Institute of Technology, Kharagpur, India-721302.
(2) Department of Mathematics, BITS, Goa Campus, India.

Abstract
The concept of duality plays an important role in optimization theory. This paper discusses some relations between primal and dual nonlinear programming problems in fuzzy environment. Here, fuzzy feasible region for a general fuzzy nonlinear programming is formed and the concept of fuzzy feasible solution is defined. First order dual relation for fuzzy nonlinear programming problem is studied.

Keywords: Fuzzy mathematical programming; Membership function; Fuzzy inequality; First order dual.

1 Introduction
In a general linear/nonlinear programming problem, if uncertainty appears in the form of fuzzy inequality or fuzzy numbers in the coefficients or fuzzy valued functions, either in the objective function or in the constraints or in both then that problem is treated as a fuzzy linear/nonlinear programming problem. Several duals for general crisp linear and nonlinear programming problems exist in literature. Duality for the fuzzy linear programming was first introduced by Rodder and Zimmerman [2]. Wu [3] studied weak and strong duality results for fuzzy linear programming problem where the coefficients are fuzzy numbers. Zhang, Yuan and Lee [5] used the concept of derivatives for convex fuzzy mapping and developed the necessary and sufficient optimality condition for the existence of solution for fuzzy mathematical programming using Lagrange dual function which are parallel to KKT condition in crisp case. Wu [4] defined fuzzy valued lagrange function and proved that the primal and dual fuzzy mathematical programming problems have

*Corresponding author. Email address: geetanjali@maths.iitkgp.ernet.in, Tel:+913222283680
no duality gap under suitable convexity assumption. Panda and Jaiswal [1] established
duality result between the fuzzy nonlinear programming and its Lagrange dual. However
very few developments are made in the context of duality for fuzzy nonlinear programming
problems to study the primal dual relationship. In this paper, Wolfe’s first order dual for
a general fuzzy nonlinear programming problem is constructed, which is different from
the approaches of Wu and Zhang, Yuan and Lee. They have considered fuzzy mappings
and fuzzy coefficients in the optimization problems, but here we have considered fuzzy
inequalities, by accepting some goals of the objective function and the constraints of the
fuzzy nonlinear programming problem.
This paper is divided into 4 sections. First section is introductory by nature. In Section 2,
feasible region for a fuzzy nonlinear programming problem is discussed. In Section 3, we
formulate the first order fuzzy dual and establish some modified duality results for fuzzy
primal and dual pairs. Section 4 discusses some future research scope.
Throughout the paper we consider the following concepts of fuzzy set theory.
Let $X$ be a universal set. Then a fuzzy set $\tilde{A}$ in $X$ is the set of ordered pairs $\tilde{A} = \{(x, \tilde{\mu}_{\tilde{A}}(x))|x \in X\}$, where $\tilde{\mu}_{\tilde{A}}(x)$ is called the membership function or grade of membership
of $x$ in $\tilde{A}$ which maps $X$ to $[0, 1]$.
In this paper we have used the fuzzy inequality due to Zimmermann [6]. The fuzzy inequal-
ity $a \preceq b; a, b \in \mathbb{R}$; where $\preceq$ is the fuzzified version of $\leq$, is a fuzzy set whose membership
function $\tilde{\mu}(a \preceq b)$ is defined by

$$\tilde{\mu}(a \preceq b) = \begin{cases} 
1 & a \leq b \\
0 & a \geq b + \alpha 
\end{cases} \in [0, 1] \quad b \leq a \leq b + \alpha,$$

where $\alpha$ is the tolerance limit for the inequality $a \preceq b$ at $b$.
Let $\odot$ be any binary operations $\oplus$ or $\otimes$ between two fuzzy sets $\tilde{A}$ and $\tilde{B}$,corresponding to
$\circ$ as $+$ and $\times$. Then from extension principle,

$$\mu_{\tilde{A} \circ \tilde{B}}(z) = \sup_{x \otimes y = z} \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}.$$ 

Also

$$\mu_{\tilde{A} \cap \tilde{B}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}.$$ 

2 Fuzzy nonlinear programming and its feasible region
Consider a general nonlinear programming problem,

$$(NP) \quad \min f(x) \quad \text{subject to} \quad g(x) \leq 0$$

where $f: \mathbb{R}^n \rightarrow \mathbb{R}, g = (g_1, g_2, ..., g_m), g_i : \mathbb{R}^n \rightarrow \mathbb{R}.$
If $f$ and $g_i$ are differentiable functions then its first order Wolfe’s dual $(ND)$ is,

$$(ND) \quad \max L(x, u), \quad \text{subject to} \quad \nabla L(x, u) = 0, \ u \geq 0, \ u \geq 0,$$
where \( L(x,u) = f(x) + u^T g(x), u \in R^m, s \in R^n. \)

In a fuzzy nonlinear programming problem, vagueness is present in the objective function or constraints or in both. Here we consider the presence of vagueness in the objective function as well as constraints of \((NP)\) in the form of fuzzy inequalities. If it is required to solve this problem where minimum value of \( f(x) \) is essentially less than \((\leq)\) some goal \( Z_0 \) which is prefixed by the decision maker and the constraints are not exactly less than equal to zero rather than essentially less than equal to \((\leq)\) zero, in that case, \((NP)\) takes the following form in fuzzy environment and called as fuzzy nonlinear programming\((FNP)\).

\[
(FNP) \quad \text{Find} \quad x \in R^n \\
\text{such that} \quad f(x) \leq Z_0 \\
g_i(x) \leq 0,
\]

where \( Z_0 \) is the aspiration level of the primal objective function. The fuzzy inequalities, \( f(x) \leq Z_0 \) and \( g_i(x) \leq 0 \) hold with certain degree of satisfaction, which can be determined from their corresponding membership functions, \( \mu_i : R \rightarrow [0,1], i = 0,1,...,m. \)

### 2.1 Feasible region for \((FNP)\)

Feasible region for crisp nonlinear programming problem is a crisp set. But, due to the presence of flexible fuzzy constraints or fuzzy objective function or both in the optimization problem, feasible region, which is the set of feasible solutions of \((FNP)\), is a fuzzy set,

\[
\tilde{D} = \{ x \in R^n \mid f(x) \leq Z_0, g_i(x) \leq 0, i = 1,2,\ldots,m \},
\]

which is the decision space corresponding to the fuzzy constraints of \((FNP)\). Its membership function, \( \mu_{\tilde{D}} : R \rightarrow [0,1] \) can be determined from the membership functions of individual fuzzy sets as,

\[
\mu_{\tilde{D}}(x) = \min\{\mu_0(f(x)), \mu_1(f_1(x)), \ldots, \mu_m(f_i(x))\}.
\]

So the crisp equivalent of \((FNP)\) becomes,

\[
(CFNP) \quad \max_{x \in \tilde{D}} \mu_{\tilde{D}}(x), \quad \mu_{\tilde{D}}(x) \leq \mu_0(f(x)), \mu_{\tilde{D}}(x) \leq \mu_1(f_1(x)), \text{ for each } i.
\]

The fuzzy feasible region, \( \tilde{D} \), of \((FNP)\) may also be interpreted as follows.

Let \( p_i (i=0,1,\ldots,m) \) are subjectively chosen nonzero positive constants of admissible violations associated with the fuzzy inequalities \((FNP)\). For some \( x \in \tilde{D} \), if \( f(x) \) fully satisfies the goal \( Z_0 \), then \( \mu_0(f(x)) = 1, \) if \( f(x) \) partially satisfies the goal within its flexibility limit, then \( 0 \leq \mu_0(f(x)) \leq 1 \) and if \( f(x) \) goes beyond the flexibility limit of the goal, then \( \mu_0(f(x)) = 0. \) Hence \( \mu_0(f(x)) \in [0,1], Z_0 \leq f(x) \leq Z_0 + p_0. \)

Similar argument can be made for the fuzzy sets \( g_i(x) \leq 0 \) and we conclude, \( \mu_i(g_i(x)) \in [0,1], 0 \leq g_i(x) \leq p_i \) for \( i = 1,2,\ldots,m. \)

If \( f(x) \leq Z_0 \) is \( \alpha\% \) satisfied and each \( g_i(x) \leq 0 \) is \( \beta_i\% \) satisfied, then the region satisfying both fuzzy events has satisfaction level \( \gamma = \min\{\alpha\%, \beta_i\%, i = 1,2,\ldots,m\}. \) This is a particular feasible region with degree of membership value \( \gamma. \) Corresponding to every \( \gamma \in [0,1], \) there exists a feasible region.
Consider the following fuzzy nonlinear programming problem

\[ FR(p_1^*, p^*) = \{ x \in \mathbb{R}^n | f(x) \leq Z_0 + p_0^0, g(x) \leq p^* \} \]

Suppose \( X \) be the universe, whose generic elements are the sets \( FR(p_i^*, p^*) \). We define a function \( \mu_{FR} : X \rightarrow [0, 1] \) by \( \mu_{FR}(FR(p_i^*, p^*)) = \min\{\mu_0^*, \mu_1^*, \mu_2^*, \ldots, \mu_m^*\} \). Then the feasible region, \( \tilde{D} \) of \((FNP)\) is equivalent to

\[ \tilde{D} = \bigcup_{(p_1^*, p^*)} \{(FR(p_1^*, p^*), \mu_{FR}(FR(p_1^*, p^*)))|FR(p_1^*, p^*) \in X\} \]

which is a fuzzy set in \( X \). A member of \( \tilde{D} \) is a set of points in \( \mathbb{R}^n \) (with same membership value) which is generated with unique aspiration level \( Z_0 + p_0^0 \) and \( p^* \) that is \( f(x) \leq Z_0 + p_0^0 \) and \( g_i(x) \leq p_i^* \). Conversely any feasible point \( x \) of \((FNP)\) corresponds to unique aspiration levels \( p_0^* \) and \( p_i^* \) for which, \( f(x) \leq Z_0 + p_0^0 \) and \( g_i(x) \leq p_i^* \) at \( x \) for each \( i = 1, \ldots, m \). That is, \( x \in FR(p_0^*, p^*) \) for some \( p_0^* \) and \( p_i^* \). This concept may be explained through the following example.

**Example 2.1.** Consider the following fuzzy nonlinear programming problem

Find \((x, y) \in \mathbb{R}^2\)

such that \(2x + 3y \leq 5\)

\(x^2 + y^2 \leq 1\)

with membership function, \( \mu_0 \) and \( \mu_1 \) defined by,

\[ \mu_0(2x + 3y) = \begin{cases} 
1 & \text{if } 2x + 3y \leq 5 \\
6 - 2x - 3y & \text{if } 5 \leq 2x + 3y \leq 6 \\
0 & \text{if } 2x + 3y \geq 6 
\end{cases} \]

and

\[ \mu_1(x^2 + y^2 - 1) = \begin{cases} 
1 & \text{if } x^2 + y^2 \leq 1 \\
2 - (x^2 + y^2) & \text{if } 1 \leq x^2 + y^2 \leq 2 \\
0 & \text{if } x^2 + y^2 \geq 2. 
\end{cases} \]

Consider the admissible violations for \(2x + 3y \leq 5\) and \(x^2 + y^2 \leq 1\) as \( p_0 = 1 \) and \( p_1 = 1 \) respectively. In particular for \( p_0^* = 0.8, p_1^* = 0.25 \), we have \( \mu_0(5.8) = 0.2 \) and \( \mu_1(0.25) = 0.75 \). Then

\[ FR(0.8, 0.25) = \{(x, y) \in \mathbb{R}^2, 2x + 3y \leq 5.8, x^2 + y^2 - 1 \leq 0.25\} \]

with membership value

\[ \mu_{FR}(FR(0.8, 0.25)) = \min\{0.2, 0.75\} = 0.2. \]
The fuzzy feasible region is,

\[ \tilde{D} = \{(x, y) \in \mathbb{R}^2, 2x + 3y \leq 5, x^2 + y^2 \leq 1\} \]

\[ = \{FR(p^*_0, p^*)| 2x + 3y \leq 5 + p^*_0, x^2 + y^2 - 1 \leq p^*_1\} \]

and

\[ \mu_{FR}(FR(p^*_0, p^*)) = \min\{\mu_0, \mu_1\}. \]

We prove the following relation between \((FNP)\) and \((CFNP)\), which is used to prove the duality results in next section.

**Theorem 2.1.** For every feasible solution of \((FNP)\) there exists \(\lambda \in [0, 1]\) such that \((x, \lambda)\) is a feasible solution of \((CFNP)\).

**Proof.** Let \(x \in \tilde{D}\). Then following the above discussion for fuzzy feasible region, we conclude that there exist some \(p^*_0 \in [0, p_0]\), \(p^*_i \in [0, p_i]\), \(i = 1, 2, \ldots, m\), such that \(x \in FR(p^*_0, p^*)\) whose membership value is \(\mu_{FR}(FR(p^*_0, p^*)) = \min\{\mu_0^*, \mu_1^*, \mu_2^*, \ldots, \mu_m^*\}\). \(x \in FR(p^*_0, p^*)\) implies

\[f(x) \leq Z_0 + p^*_0, \ g(x) \leq p^*.\]

For some \(\lambda_0, \lambda_i \in [0, 1]\), let \(p^*_0 = (1 - \lambda_0)p_0\) and \(p^*_i = (1 - \lambda_i)p_i\), \(i = 1, 2, \ldots, m\).

Hence the above inequalities become

\[(\lambda_0 - 1)p_0 \leq Z_0 - f(x), \ (\lambda_i - 1)p_i \leq -g_i(x), \ i = 1, 2, \ldots, m.\]

For \(\lambda = \min(\lambda_0, \lambda_i)\) we have,

\[(\lambda - 1)p_0 \leq Z_0 - f(x), \ (\lambda - 1)p_i \leq -g_i(x).\]

Hence \((x, \lambda)\) is a feasible solution of \((CFNP)\). \(\square\)

### 3 First order dual of \((FNP)\)

Since the functions \(f\) and \(g_i\) have vagueness due to the presence of the fuzzy inequality “essentially less than”, so the Lagrange function \(L(x, u) = f(x) + u^T g(x)\) have certain degree of vagueness. It is required to maximize \(L(x, u)\) while formulating its dual. In case, the decision maker agrees to accept the maximum of \(L(x, u)\) essentially greater than equal to some goal \(W_0\), with subjectively chosen nonzero positive constant, \(q_0\), of the admissible violation of the fuzzy constraint \(L(x, u) \geq W_0\), then the objective function can be treated as \(L(x, u) \geq W_0\), where “\(\geq\)” means “essentially greater than equal to”, which is a linguistic relation with degree of satisfaction \(\eta_0(L(x, u))\), whose value is zero of if this fuzzy inequality is not satisfied, 1 if it is fully satisfied, and lies in \([0, 1]\) if it is partially satisfied. Hence

\[ \eta_0(L(x, u)) = \begin{cases} 1 & L(x, u) \geq W_0 \\ \in [0, 1] & W_0 - q_0 \leq L(x, u) \leq W_0 \\ 0 & L(x, u) \leq W_0 - q_0. \end{cases} \]
In the light of (ND), the corresponding Wolfe’s dual in fuzzy sense and its crisp equivalent takes the following form.

\[(FND) \ \text{Find}\ (x, u) \in R^n \times R^m \ \text{subject to} \ L(x, u) \geq W_0, \ \nabla L(x, u) = 0, \ u \geq 0.\]

\[(CFND) \ \max_{(x, u) \in R^n \times R^m} \eta_0(L(x, u)) \ \text{subject to} \ \nabla L(x, u) = 0, \ u \geq 0.\]

From well known duality relation between (NP) and (ND), it is true that if \(x_0\) is a feasible solution of (NP) and \(u_0\) is a feasible solution of (ND), then \(f(x_0) \geq L(x_0, u_0)\) and (ND) is a strong dual of (NP) if it is a convex programming problem. Here we see the relation between (FNP) and (FND) in terms of the goals \(Z_0\) and \(W_0\).

**Theorem 3.1.** Suppose \(x_0\) is a feasible solution of (NP) and \(u_0\) is a feasible solution of ND. If the corresponding objective value of (NP) fully satisfies the goal \(Z_0\) then the weak duality between (FNP) and (FND) holds. That is \(Z_0 \geq W_0\). If the corresponding objective value of (NP) partially satisfies the goal \(Z_0\) then the weak duality between (FNP) and (FND) partially holds. That is \(Z_0 \geq W_0 - (p_0 + q_0)\).

**Proof.** By weak duality result between (NP) and (ND), \(f(x_0) \geq L(x_0, u_0)\).
Since \(x_0\) is the feasible solution of (NP), so \(g_i(x_0) \leq 0\). Hence
\[\mu_i(g_i(x_0)) = 1, \quad i = 1, 2, \ldots, m.\]

If \(f(x_0)\) fully satisfies the goal \(Z_0\), then \(f(x_0) \leq Z_0\). Hence \(\mu_0(f(x_0)) = 1\).

\[\eta(L(x_0, u_0))) = \eta(f(x_0) + u_0^Tg(x_0))\]
\[= \min\{\mu_0(f(x_0)), \mu_1(g_1(x_0)), \ldots, \mu_m(g_m(x_0))\}\]

So \(L(x_0, u_0) \geq W_0\). Combining all results we have \(Z_0 \geq f(x_0) \geq L(x_0, u_0) \geq W_0\).

If \(f(x_0)\) partially satisfies the goal \(Z_0\), then \(Z_0 \leq f(x_0) \leq Z_0 + p_0\). Hence \(\mu_0(f(x_0)) \in [0, 1]\).

\[\eta(L(x_0, u_0))) = \eta(f(x_0) + u_0^Tg(x_0))\]
\[= \min\{\mu_0(f(x_0)), \mu_1(g_1(x_0)), \ldots, \mu_m(g_m(x_0))\}\]

So \(W_0 - q_0 \leq L(x_0, u_0) \leq W_0\). Combining above results we have
\[Z_0 + p_0 \geq f(x_0) \geq L(x_0, u_0) \geq W_0 - q_0.\]

Hence \(Z_0 \geq W_0 - (p_0 + q_0)\).

**Theorem 3.2.** Suppose \(x_0\) is a feasible solution of (FNP) and \(u_0\) is a feasible solution of (FND). If \(f\) and \(g\) are differentiable convex functions at \(x_0\) and the degree of satisfaction of the fuzzy inequalities in both (FNP) and (FND) are linear functions then there exists a positive number \(K\) such that \(W_0 \leq Z_0 + K\).

**Proof.** The membership functions \(\mu_0(f(x)), \mu_i(g_i(x))\) and \(\eta_0(L(x, u))\) are 1 if the respective fuzzy inequalities \(f(x) \leq Z_0, g_i(x) \leq 0\) and \(L(x, u) \geq W_0\) are fully satisfied, and are 0 if
they violate their flexibility bound. In addition to this, the degree of satisfaction of the inequalities are linear functions. Hence the corresponding membership functions become,

\[ \mu_f(x) = \begin{cases} 
1 & f(x) \leq Z_0 \\
1 - \frac{f(x) - Z_0}{p_0} & Z_0 \leq f(x) \leq Z_0 + p_0 \\
0 & f(x) \geq Z_0 + p_0
\end{cases} \]

\[ \mu_i(g_i(x)) = \begin{cases} 
1 & g_i(x) \leq 0 \\
1 - \frac{g_i(x)}{p_i} & 0 \leq g_i(x) \leq p_i \\
0 & g_i(x) \geq p_i
\end{cases} \]

\[ \eta_0(L(x, u)) = \begin{cases} 
1 & L(x, u) \geq W_0 \\
1 - (W_0 - L(x, u))/q_0 & W_0 - q_0 \leq L(x, u) \leq W_0 \\
0 & L(x, u) \leq W_0 - q_0.
\end{cases} \]

Using the above membership functions, \((CFNP)\) and \((CFND)\) becomes,

\[(CFNP) \quad \text{max} \quad \lambda \]

subject to \((\lambda - 1)p_0 + f(x) \leq Z_0 \]

\[(\lambda - 1)p_i + g_i(x) \leq 0, \quad i = 1, 2, \ldots, m, \quad 0 \leq \lambda \leq 1\]

\[(CFND) \quad \text{max} \quad \eta \]

subject to \((\eta - 1)q_0 \leq L(x, u) - W_0 \]

\[\nabla L(x, u) = 0, \quad u \geq 0, \quad 0 \leq \eta \leq 1.\]

Since \(x_0\) is a feasible solution of \((FNP)\), there exists \(0 \leq \lambda_0 \leq 1\) such that \((x_0, \lambda_0)\) is the feasible solution of \((CFNP)\). Also since \(u_0\) is feasible solution of \((FND)\), there exists \(0 \leq \eta_0 \leq 1\) such that \((x_0, u_0, \eta_0)\) is the feasible solution of \((CFND)\). Hence

\[(\lambda_0 - 1)p_0 \leq Z_0 - f(x_0) \quad \text{(3.1)}\]

\[(\lambda_0 - 1)p \leq -g(x_0) \quad \text{(3.2)}\]

\[(\eta_0 - 1)q_0 \leq L(x_0, u_0) - W_0 \quad \text{(3.3)}\]

\[\nabla f(x_0) + u_0^T \nabla g(x_0) = 0 \quad \text{(3.4)}\]

Adding Inequalities (3.1), (3.2) and (3.3),

\[(\lambda_0 - 1)p_0 + (\lambda_0 - 1)u_0^T p + (\eta_0 - 1)q_0 \leq Z_0 - (f(x_0) + u_0^T g(x_0)) + L(x_0, u_0) - W_0 \]

\[= Z_0 - W_0 + f(x_0) - f(x_0) \]

\[+ u_0^T (g(x_0) - g(x_0)) \quad \text{(3.5)}\]
Since $f$ and $g$ are convex and differentiable at $x_0$, so
\[ f(x_0) - f(x) \geq (\nabla f(x))^T (x_0 - x) \]
and
\[ g(x_0) - g(x) \geq (\nabla g(x))^T (x_0 - x). \]
Hence right hand side of Inequality (3.5) becomes,
\[ Z_0 - W_0 - (\nabla f(x_0) + u_0^T \nabla g(x_0))^T (x_0 - x_0). \]
Using Eq. (3.4), Inequality (3.5) reduces to
\[(\lambda_0 - 1)p_0 + (\lambda_0 - 1)u_0^T p + (\eta_0 - 1)q_0 \leq Z_0 - W_0. \]
Let $K = (1 - \lambda_0)p_0u_0^T p + (1 - \eta_0)q_0$. Then $K \geq 0$. Hence $Z_0 \geq W_0 - K$. \qed

4 Conclusion

In this paper, duality relation between the fuzzy primal nonlinear programming problem and the fuzzy version of Wolfe’s first order dual are studied. We may observe that if the fuzzy inequalities of ($FNP$) are fully satisfied then weak duality between ($FNP$) and ($FND$) holds($Z_0 \geq W_0$), which is parallel to the weak duality concept between ($NP$) and ($ND$). But this is not true in case the fuzzy inequalities are partially satisfied. In that case the duality gap is expressed in terms of the tolerance levels, $p_0, p_i$ of the fuzzy inequalities.

Also if the fuzzy nonlinear programming problem is defined in other fuzzy environments, in the presence of fuzzy valued functions or in the presence of fuzzy coefficients, then the modified weak duality results may not follow directly. Several type of duals like higher order dual, symmetric dual etc. for a general nonlinear programming problem exist. We have studied the weak duality result between fuzzy primal dual pair. In the light of the concept of this paper it is possible to construct other type duals in fuzzy environment develop strong duality result between these pairs.

References


