Fuzzy $B$-subalgebras of $B$-algebra with Respect to $t$-norm

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Abstract
In this paper, we apply the concept of $t$-norm $T$ to fuzzy structure of $B$-algebras. The notion of a fuzzy $B$-subalgebra of $B$-algebras with respect to $t$-norm is introduced and several related properties are investigated. The direct product and $T$-product of $T$-fuzzy subalgebra of $B$-algebra is investigated.

Keywords: $B$-algebra, $t$-norm, $T$-fuzzy subalgebra, homomorphism, upper level cuts.

AMS Mathematics Subject Classification (2010): 06F35, 03G25, 94D05

1 Introduction

$BCK$-algebras and $BCI$-algebras are two important classes of logical algebras introduced by Imai and Iseki [8-10]. It is known that the class of $BCK$-algebra is a proper subclass of the class of $BCI$-algebras. Hu and Li [7] and Iseki [6] introduced a wide class of abstract algebras: $BCH$-algebras. They have shown that the class of $BCI$-algebra is a proper subclass of the $BCH$-algebras. Neggers and Kim [17, 18] introduced a new notion, called a $B$-algebras which is related to several classes of algebras of interest such as $BCH/BCI/BCK$-algebras. Cho and Kim [3] discussed further relations between $B$-algebras and other topics especially quasi-groups. Park and Kim [19] obtained that every quadratic $B$-algebra on a field $X$ with $|X| \geq 3$ is a $BCI$-algebra. The concept of fuzzy sets was introduced by Zadeh [27]. At present these ideas have been applied to other algebraic structures such as semigroups, groups [4, 20] rings, ideals, modules, vector spaces, etc. In 2002, Jun et al. [11] applied the concept of fuzzy sets to $B$-algebras. In 2003, Ahn and

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Triangular norms ($t$-norms for short) were introduced by Schweizer and Sklar in [21-24], following some ideas of Menger in the context of probabilistic metric spaces [16] (as statistical metric spaces were called after 1964). With the development of $t$-norms in statistical metric spaces, they also play an important role in decision making, in statistics as well as in the theories of cooperative games. In particular, in fuzzy set theory, $t$-norms have been widely used for fuzzy operations, fuzzy logic and fuzzy relation equations [26]. In recent years, a systematic study concerning the properties and related matters of $t$-norms have been made by Klement et al. [12-14]. In the present paper, the fuzzy $B$-subalgebra of the $B$-algebras with respect to a $t$-norm $T$ is redefined and hence generalize the notion in [2, 11] and obtained some of their properties. Also the direct product and $T$-product of $T$-fuzzy subalgebra of $B$-algebra is subalgebras are introduced and discussed their properties in detail.

2 Preliminaries

In this section, some elementary aspects that are necessary for this paper are included. An algebra $(X; *, 0)$ of type $(2, 0)$ is called a B-algebra [17] if it satisfies the following axioms:

$B1. \quad x * x = 0$

$B2. \quad x * 0 = x$

$B3. \quad (x * y) * z = x * (z * (0 * y)), \text{ for all } x, y \in X.$

Example 2.1. [17], Let $X$ be the set of all real numbers except for a negative integer $-n$. Define a binary operation $*$ on $X$ by $x * y = \frac{n(x-y)}{n+y}$. Then $(X, *, 0)$ is a B-algebra.

A non-empty subset $S$ of a B-algebra $X$ is called a B-subalgebra ([18]) of $X$ if $x * y \in S$ for any $x, y \in S$. A partial ordering $\leq$ on $X$ can be defined by $x \leq y$ if and only if $x * y = 0$. 

Definition 2.1. [16], A triangular norm ($t$-norm) is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ that satisfies:

$(T1)$ boundary condition: $T(x, 1) = x$;

$(T2)$ commutativity: $T(x, y) = T(y, x)$;

$(T3)$ associativity: $T(x, T(y, z)) = T(T(x, y), z)$;

$(T4)$ monotonicity: $T(x, y) \leq T(x, z)$ whenever $y \leq z$, for all $x, y, z \in [0, 1]$.

Some example [14] of $t$-norms are the minimum $T_M(x, y) = \min(x, y)$, the product $T_P(x, y) = x \cdot y$ and the Lukasiewicz $t$-norm $T_L(x, y) = \max(x + y - 1, 0)$ for all $x, y \in [0, 1]$. Also it is well known [5, 13] that if $T$ is a $t$-norm, then $T(x, y) \leq \min\{x, y\}$ for all $x, y \in [0, 1]$. 

We now review some fuzzy logic concepts as follows:

Let $X$ be the collection of objects denoted generally by $x$ then a fuzzy set $A$ in $X$ is defined as $A = \{< x, \alpha_A(x) > : x \in X\}$ where $\alpha_A(x)$ is called the membership value of $x$ in $A$ and $0 \leq \alpha_A(x) \leq 1$. For any fuzzy sets $A$ and $B$ of a set $X$, we define

$A \cap B = \min\{\alpha_A(x), \alpha_B(x)\}$ \quad \forall x \in X.$

Let $f$ be a mapping from the set $X$ into the set $Y$. Let $B$ be a fuzzy set in $Y$. Then the inverse image of $B$, denoted by $f^{-1}(B)$ in $X$ and is given by $f^{-1}(\alpha_B)(x) = \alpha_B(f(x))$. 

Conversely, let \( A \) be a fuzzy set in \( X \) with membership function \( \alpha_A \). Then the inverse image of \( A \), denoted by \( f(A) \) in \( Y \) and is given by

\[
\alpha_{f(A)}(y) = \begin{cases} 
\sup_{x \in f^{-1}(y)} \alpha_A(x), & \text{if } f^{-1}(y) \neq \emptyset \\
1, & \text{otherwise.}
\end{cases}
\]

**Definition 2.2.** Let \( P \) be a \( t \)-norm. Denote by \( \triangle_P \) the set of elements \( x \in [0, 1] \) such that \( P(x, x) = x \), that is,

\[
\triangle_P := \{ x \in [0, 1] | P(x, x) = x \}.
\]

A fuzzy set \( A \) in \( X \) is said to satisfy imaginable property with respect to \( P \) if \( \text{Im}(\alpha_A) \subseteq \triangle_P \).

### 3 T-fuzzy subalgebra of B-algebra

In this section, \( T \)-fuzzy subalgebra of \( B \)-algebra are defined and some propositions and theorems are presented. In what follows, let \( X \) denote a \( B \)-algebra unless otherwise specified.

**Definition 3.1.** Let \( A \) be a fuzzy set in \( X \). Then the set \( A \) is \( T \)-fuzzy subalgebra over the binary operator \( * \) if it satisfies (TS1) \( \alpha_A(x * y) \geq T\{\alpha_A(x), \alpha_A(y)\} \) for all \( x, y \in X \).

Let us illustrate this definition using the following example.

**Example 3.1.** Let \( X = \{0, 1, 2, 3, 4, 5\} \) be a \( B \)-algebra (see [18], example 3.5) with the following Cayley table:

<table>
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<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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</table>

Let \( T_m : [0, 1] \times [0, 1] \to [0, 1] \) be functions defined by \( T_m(x, y) = \max(x + y - 1, 0) \) for all \( x, y \in [0, 1] \). Then \( T_m \) is a \( t \)-norm. Define a fuzzy set \( A \) in \( X \) by \( \alpha_A(0) = 0.8, \alpha_A(1) = \alpha_A(2) = 0.72 \) and \( \alpha_A(x) = 0.47 \) for all \( x \in X \setminus \{0, 1, 2\} \). Then \( A \) is a \( T_m \)-normed fuzzy subalgebra of \( X \).

**Definition 3.2.** A \( T \)-fuzzy subalgebra \( A \) is called an imaginable \( T \)-fuzzy subalgebra of \( X \) if \( \alpha_A \) satisfy the imaginable property with respect to \( T \).

**Example 3.2.** Consider \( T_m \) be a \( t \)-norm and \( X = \{0, 1, 2, 3, 4, 5\} \) be a \( B \)-algebra in Example 3.1. Define a fuzzy set \( A \) in \( X \) by \( \alpha_A(x) = 1 \), if \( x \in \{0, 1, 2\} \) and \( \alpha_A(x) = 0 \), if \( x \in X \setminus \{0, 1, 2\} \). It is easy to check that \( \alpha_A(x * y) \geq T_m\{\alpha_A(x), \alpha_A(y)\} \) for all \( x, y \in X \). Also \( \text{Im}(\alpha_A) \subseteq \triangle_{T_m} \). Hence \( A \) is an imaginable \( T_m \)-fuzzy subalgebra of \( X \).

**Proposition 3.1.** If \( A \) is an imaginable \( T \)-fuzzy subalgebra of \( X \), then \( \alpha_A(0) \geq \alpha_A(x) \) for all \( x \in X \).

**Proof.** Let \( x \in X \). Then \( \alpha_A(0) = \alpha_A(x * x) \geq T\{\alpha_A(x), \alpha_A(x)\} = \alpha_A(x) \).

\[ \square \]
Proposition 3.2. Let $A$ be an imaginable $T$-fuzzy subalgebra of $X$ and let $n \in \mathbb{N}$ (the set of natural numbers). Then

(i) $\alpha_A(\prod_{i=1}^n x \ast x) \geq \alpha_A(x)$, for any odd number $n$,

(ii) $\alpha_A(\prod_{i=1}^n x \ast x) = \alpha_A(x)$, for any even number $n$.

Proof. Let $x \in X$ and assume that $n$ is odd. Then $n = 2p - 1$ for some positive integer $p$. We prove the proposition by induction. By Proposition 3.1 $\alpha_A(0) \geq \alpha_A(x)$ for all $x \in X$. Suppose that $\alpha_A(\prod_{i=1}^{2p-1} x \ast x) \geq \alpha_A(x)$. Then by assumption,

$$\alpha_A(\prod_{i=1}^{2(p+1)-1} x \ast x) = \alpha_A(\prod_{i=1}^{2p+1} x \ast x) = \alpha_A(\prod_{i=1}^{2p-1} x \ast (x \ast x)) = \alpha_A(\prod_{i=1}^{2p-1} x \ast x) \geq \alpha_A(x)$$

which proves (i). Proof are similar for the case (ii). $\square$

Proposition 3.3. If $A$ is an imaginable $T$-fuzzy subalgebra of $X$, then $\alpha_A(0 \ast x) \geq \alpha_A(x)$ for all $x \in X$.

Proof. For all $x \in X$,

$$\alpha_A(0 \ast x) \geq T\{\alpha_A(0), \alpha_A(x)\} = T\{\alpha_A(x \ast x), \alpha_A(x)\} \geq T\{\{\alpha_A(x), \alpha_A(x)\}, \alpha_A(x)\} = \alpha_A(x)$$

since it is imaginable. This completes the proof. $\square$

Theorem 3.1. Let $A$ be an imaginable $T$-fuzzy subalgebra of $X$. If there exists a sequence $x_n$ in $X$ such that $\lim_{n \to \infty} \alpha_A(x_n) = 1$ then $\alpha_A(0) = 1$.

Proof. By Proposition 3.1, $\alpha_A(0) \geq \alpha_A(x)$ for all $x \in X$, therefore $\alpha_A(0) \geq \alpha_A(x_n)$ for every positive integer $n$. Consider, $1 \geq \alpha_A(0) \geq \lim_{n \to \infty} \alpha_A(x_n) = 1$. Hence, $\alpha_A(0) = 1$. $\square$

The intersection of any two $T$-fuzzy subalgebras is also a $T$-fuzzy subalgebra, which is proved in the following theorem.

Theorem 3.2. Let $A_1$ and $A_2$ be two $T$-fuzzy subalgebras of $X$. Then $A_1 \cap A_2$ is a $T$-fuzzy subalgebra of $X$.

Proof. Let $x, y \in A_1 \cap A_2$. Then $x, y \in A_1$ and $A_2$. Now,

$$\alpha_{A_1 \cap A_2}(x \ast y) = \min\{\alpha_{A_1}(x \ast y), \alpha_{A_2}(x \ast y)\},$$

$$\geq \min\{T\{\alpha_{A_1}(x), \alpha_{A_1}(y)\}, T\{\alpha_{A_2}(x), \alpha_{A_2}(y)\}\}$$

$$\geq T\{\min\{\alpha_{A_1}(x), \alpha_{A_2}(x)\}, \min\{\alpha_{A_1}(y), \alpha_{A_2}(y)\}\}$$

$$= T\{\alpha_{A_1 \cap A_2}(x), \alpha_{A_1 \cap A_2}(y)\}$$

Hence, $A_1 \cap A_2$ is a $T$-fuzzy subalgebra of $X$. $\square$

The above theorem can be generalized as follows.

Theorem 3.3. Let $\{A_i| i = 1, 2, 3, 4, \ldots\}$ be a family of $T$-fuzzy subalgebra of $X$. Then $\bigcap A_i$ is also a $T$-fuzzy subalgebra of $X$, where $\bigcap A_i = \{x, \min \alpha_{A_i}(x) > x : x \in X\}$. 

The set \( \{x \in X : \alpha_A(x) = \alpha_A(0)\} \) is denoted by \( I_{\alpha_A} \). The set \( I_{\alpha_A} \) is also subalgebra of \( X \).

**Theorem 3.4.** Let \( A \) be a \( T \)-fuzzy subalgebra of \( X \), then the set \( I_{\alpha_A} \) is a subalgebra of \( X \).

**Proof.** Let \( x, y \in I_{\alpha_A} \). Then \( \alpha_A(x) = \alpha_A(0) = \alpha_A(y) \) and so, \( \alpha_A(x \ast y) \geq T\{\alpha_A(x),\alpha_A(y)\} = T\{\alpha_A(0),\alpha_A(0)\} = \alpha_A(0) \). By using Proposition 3.1, we know that \( \alpha_A(x \ast y) \leq \alpha_A(0) \). Hence \( \alpha_A(x \ast y) = \alpha_A(0) \) or equivalently \( x \ast y \in I_{\alpha_A} \). Therefore, the set \( I_{\alpha_A} \) is subalgebra of \( X \).

As is well known, the characteristic function of a set is a special fuzzy set. Suppose \( A \) is a non-empty subset of \( X \). By \( \chi_A \) we denote the characteristic function of \( A \), that is,

\[
\chi_A(x) = \begin{cases} 
1, & \text{if } x \in A \\
0, & \text{otherwise.}
\end{cases}
\]

**Theorem 3.5.** If \( A \) is a \( B \)-subalgebra of \( X \), then the characteristic function \( \chi_A \) is a \( T \)-fuzzy subalgebra of \( X \).

**Proof.** Let \( x, y \in X \). We consider here four cases:

**Case (i)** If \( x, y \in A \) then \( x \ast y \in A \) since \( A \) is a \( B \)-subalgebra of \( X \). Then

\[
\chi_A(x \ast y) = 1 \geq T\{\chi_A(x),\chi_A(y)\}.
\]

**Case (ii)** If \( x, y \notin A \), then \( \chi_A(x) = 0 = \chi_A(y) \). Thus

\[
\chi_A(x \ast y) = 0 = \min\{0,0\} \geq T\{0,0\} = T\{\chi_A(x),\chi_A(y)\}.
\]

**Case (iii)** If \( x \in A \) and \( y \notin A \) then \( \chi_A(x) = 1, \chi_A(y) = 0 \). Thus

\[
\chi_A(x \ast y) = 0 = T\{0,1\} = T\{1,0\} = T\{\chi_A(x),\chi_A(y)\}.
\]

**Case (iv)** If \( x \notin A \) and \( y \in A \) then by the same argument as in Case (iii), we conclude that \( \chi_A(x \ast y) \geq T\{\chi_A(x),\chi_A(y)\} \). Therefore, the characteristic function \( \chi_A \) is a \( T \)-fuzzy subalgebra of \( X \). \( \square \)

**Theorem 3.6.** Let \( A \) be a non-empty subset of \( X \). If \( \chi_A \) satisfies (TS1), then \( A \) is a \( B \)-subalgebra of \( X \).

**Proof.** Suppose that \( \chi_A \) satisfy (TS1). Let \( x, y \in A \). Then it follows from (TS1) that \( \chi_A(x \ast y) \geq T\{\chi_A(x),\chi_A(y)\} = T\{1,1\} = 1 \) so that \( \chi_A(x \ast y) = 1 \), i.e., \( x \ast y \in A \). Hence, \( A \) is a \( B \)-subalgebra of \( X \). \( \square \)

**Proposition 3.4.** Let \( P \) be a \( B \)-subalgebra of \( X \) and \( A \) be a fuzzy set in \( X \) defined by

\[
\alpha_A(x) = \begin{cases} 
\lambda, & \text{if } x \in P \\
\tau, & \text{otherwise}
\end{cases}
\]

for all \( \lambda, \tau \in [0,1] \) with \( \lambda \geq \tau \). Then \( A \) is an \( T_m \)-fuzzy subalgebra of \( X \), where \( T_m \) is the \( t \)-norm in Example 3.1. In particular if \( \lambda = 1 \) and \( \tau = 0 \) then \( A \) is an imaginable \( T_m \)-fuzzy subalgebra of \( X \). Moreover, \( I_{\alpha_A} = P \).
Proof. Let \( x, y \in X \). We consider here three cases:

**Case (i)** If \( x, y \in P \) then
\[
T_m(\alpha_A(x), \alpha_A(y)) = T_m(\lambda, \lambda)
= \max(2\lambda - 1, 0)
= \begin{cases} 
2\lambda - 1 & \text{if } \lambda \geq \frac{1}{2} \\
0 & \text{otherwise}
\end{cases}
\leq \lambda
= \alpha_A(x * y).
\]

**Case (ii)** If \( x \in P \) and \( y \notin P \) (or, \( x \notin P \) and \( y \in P \)) then
\[
T_m(\alpha_A(x), \alpha_A(y)) = T_m(\lambda, \tau)
= \max(\lambda + \tau - 1, 0)
= \begin{cases} 
\lambda + \tau - 1 & \text{if } \lambda + \tau \geq 1 \\
0 & \text{otherwise}
\end{cases}
\leq \tau
= \alpha_A(x * y).
\]

**Case (iii)** If \( x, y \notin P \) then
\[
T_m(\alpha_A(x), \alpha_A(y)) = T_m(\tau, \tau)
= \max(2\tau - 1, 0)
= \begin{cases} 
2\tau - 1 & \text{if } \tau \geq \frac{1}{2} \\
0 & \text{otherwise}
\end{cases}
\leq \tau
= \alpha_A(x * y).
\]

Hence, \( A \) is an \( T_m \)-fuzzy subalgebra of \( X \).

Assume that \( \lambda = 1 \) and \( \tau = 0 \). Then \( T_m(\lambda, \lambda) = \max(\lambda + \lambda - 1, 0) = 1 = \lambda \) and \( T_m(\tau, \tau) = \max(\tau + \tau - 1, 0) = 0 = \tau \). Thus, \( \lambda, \tau \in \triangle_{T_m} \) i.e., \( \text{Im}(\alpha_A) \subseteq \triangle_{T_m} \). So, \( A \) is an imaginable \( T_m \)-fuzzy subalgebra of \( X \).

Also,
\[
I_{\alpha_A} = \{ x \in X, \alpha_A(x) = \alpha_A(0) \} = \{ x \in X, \alpha_A(x) = \lambda \} = P
\]

Therefore, \( I_{\alpha_A} = P \). \( \square \)

**Definition 3.3.** [4], Let \( A \) is a fuzzy set in \( X \). For \( \tilde{s} \in [0,1] \), the set \( U(\alpha_A : \tilde{s}) = \{ x \in X : \alpha_A(x) \geq \tilde{s} \} \) is called upper \( \tilde{s} \)-level of \( A \).
Theorem 3.7. Let $A$ be a $T$-fuzzy subalgebra of $X$ and $\tilde{s} \in [0,1]$. Then if $\tilde{s} = 1$, the upper level set $U(\alpha_A : \tilde{s})$ is either empty or a $B$-subalgebra of $X$.

Proof. Let $\tilde{s} = 1$ and $x, y \in U(\alpha_A : \tilde{s})$. Then $\alpha_A(x) \geq \tilde{s} = 1$ and $\alpha_A(y) \geq \tilde{s} = 1$. It follows that $\alpha_A(x * y) \geq T(\alpha_A(x), \alpha_A(y)) \geq T(1, 1) = 1$ so that $x * y \in U(\alpha_A : \tilde{s})$. Hence, $U(\alpha_A : \tilde{s})$ is a $B$-subalgebra of $X$ when $s = 1$.

Theorem 3.8. If $A$ is an imaginable $T$-fuzzy subalgebra of $X$, then the upper $\tilde{s}$-level of $A$ is $B$-subalgebras of $X$.

Proof. Assume that $x, y \in U(\alpha_A : \tilde{s})$. Then $\alpha_A(x) \geq \tilde{s}$ and $\alpha_A(y) \geq \tilde{s}$. It follows that $\alpha_A(x * y) \geq T(\alpha_A(x), \alpha_A(y)) \geq T(\tilde{s}, \tilde{s}) = \tilde{s}$ so that $x * y \in U(\alpha_A : \tilde{s})$. Hence, $U(\alpha_A : \tilde{s})$ is a $B$-subalgebra of $X$.

Theorem 3.9. Let $A$ be a fuzzy set in $X$ such that the set $U(\alpha_A : \tilde{s})$ is $B$-subalgebras of $X$ for every $\tilde{s} \in [0,1]$. Then $A$ is a $T$-fuzzy subalgebra of $X$.

Proof. Let for every $\tilde{s} \in [0,1], U(\alpha_A : \tilde{s})$ is subalgebra of $X$. In contrary, let $x_0, y_0 \in X$ be such that $\alpha_A(x_0*y_0) < T(\alpha_A(x_0), \alpha_A(y_0))$. Let us consider,

$$\tilde{s}_0 = \frac{1}{2} \left[ \alpha_A(x_0 * y_0) + T(\alpha_A(x_0), \alpha_A(y_0)) \right].$$

Then $\alpha_A(x_0 * y_0) < \tilde{s}_0 \leq T(\alpha_A(x_0), \alpha_A(y_0)) \leq \min\{\alpha_A(x_0), \alpha_A(y_0)\}$ and so $x_0 * y_0 \notin U(\alpha_A : \tilde{s})$ but $x_0, y_0 \in U(\alpha_A : \tilde{s})$. This is a contradiction and hence $\alpha_A$ satisfies the inequality $\alpha_A(x * y) \geq T(\alpha_A(x), \alpha_A(y))$ for all $x, y \in X$.

Definition 3.4. [25], A mapping $f : X \rightarrow Y$ of $B$-algebra is called a homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Note that if $f : X \rightarrow Y$ is a $B$-homomorphism, then $f(0) = 0$.

Theorem 3.10. Let $f : X \rightarrow Y$ be a homomorphism of $B$-algebras. If $B = \{< x, \alpha_B(x) > : x \in X\}$ is a $T$-fuzzy $B$-subalgebra of $Y$, then the pre-image $f^{-1}(B) = \{< x, f^{-1}(\alpha_B)(x) > : x \in X\}$ of $B$ under $f$ is a $T$-fuzzy $B$-subalgebra of $X$.

Proof. Assume that $B$ is a $T$-fuzzy $B$-subalgebra of $Y$ and let $x, y \in X$. Then

$$f^{-1}(\alpha_B)(x * y) = \alpha_B(f(x * y))$$

$$= \alpha_B(f(x) * f(y))$$

$$\geq T(\alpha_B(f(x), \alpha_B(f(y))))$$

$$= T(f^{-1}(\alpha_B)(x), f^{-1}(\alpha_B)(y)).$$

Therefore $f^{-1}(B)$ is a $T$-fuzzy $B$-subalgebra of $X$.

Theorem 3.11. [18], Let $f : X \rightarrow Y$ be a homomorphism from a $B$-algebra $X$ onto a $B$-algebra $Y$. If $A = \{< x, \alpha_A(x) > : x \in X\}$ is a fuzzy subalgebra of $X$, then the image $f(A) = \{< x, f_{\sup}(\alpha_A)(x) > : x \in Y\}$ of $A$ under $f$ is a fuzzy subalgebra of $Y$.

Theorem 3.12. Let $f : X \rightarrow Y$ be a homomorphism from a $B$-algebra $X$ onto a $B$-algebra $Y$. If $A$ is an imaginable $T$-fuzzy subalgebra of $X$, then the image $f(A)$ of $A$ under $f$ is a $T$-fuzzy subalgebra of $Y$. 


Proof. Let \( A \) be an imaginable \( T \)-fuzzy subalgebra of \( X \). By Theorem 3.8 \( U(\alpha_A : \tilde{s}) \) is \( B \)-subalgebra of \( X \) for every \( \tilde{s} \in [0,1] \). Therefore by Theorem 3.11 \( f(U(\alpha_A : \tilde{s})) \) is \( B \)-subalgebras of \( Y \). But \( f(U(\alpha_A : \tilde{s})) = U(f(\alpha_A : \tilde{s})) \). Hence \( U(f(\alpha_A : \tilde{s})) \) is \( B \)-subalgebras of \( X \) for every \( \tilde{s} \in [0,1] \). By Theorem 3.9 \( f(A) \) is a \( T \)-fuzzy \( B \)-subalgebra of \( Y \).

4 Product of \( T \)-fuzzy subalgebras

In this section, the direct product and \( T \)-normed product of fuzzy subalgebras of \( B \)-algebras with respect to \( t \)-norm and \( s \)-norm are presented and several properties are studied. Before going into the product of fuzzy subalgebras of \( B \)-algebras, we first define some kind of product of fuzzy subsets.

Definition 4.1. Let \( A_1 = \{ < x, \alpha_{A_1}(x) : x \in X \} \) and \( A_2 = \{ < x, \alpha_{A_2}(x) : x \in X \} \) be fuzzy subsets of a \( B \)-algebra \( X \). Then the \( T \)-product of \( A_1 \) and \( A_2 \) denoted by \([A_1,A_2]_T = \{ < x,[\alpha_{A_1},\alpha_{A_2}]_T(x) : x \in X \} \) and is defined by \([\alpha_{A_1},\alpha_{A_2}]_T(x) = T(\alpha_{A_1}(x),\alpha_{A_2}(x)) \) for all \( x \in X \).

Theorem 4.1. Let \( A_1 \) and \( A_2 \) be two \( T \)-fuzzy subalgebra of \( X \). If \( T^* \) is a \( T \)-norm which dominates \( T \), i.e., \( T^*(T(a,b),T(c,d)) \geq T(T^*(a,c),T^*(b,d)) \) for all \( a,b,c \) and \( d \in [0,1] \), then the \( T^* \)-product of \( A_1 \) and \( A_2 \), \([A_1,A_2]_{T^*} \) is a \( T^* \)-fuzzy subalgebra of \( X \).

Proof. For any \( x,y \in X \), we have
\[
[A_1,A_2]_{T^*}(x \ast y) = T^*(\alpha_{A_1}(x \ast y),\alpha_{A_2}(x \ast y)) \\
\geq T^*(T(\alpha_{A_1}(x),\alpha_{A_1}(y)),T(\alpha_{A_2}(x),\alpha_{A_2}(y))) \\
\geq T(T^*(\alpha_{A_1}(x),\alpha_{A_1}(y)),T^*(\alpha_{A_2}(y),\alpha_{A_2}(y))) \\
= T(\alpha_{A_1},\alpha_{A_2})_{T^*}(x),T(\alpha_{A_1},\alpha_{A_2})_{T^*}(y)).
\]

Hence, \([A_1,A_2]_{T^*} \) is an \( T \)-fuzzy subalgebra of \( X \). 

Let \( f : X \to Y \) be an epimorphism of \( B \)-algebras. Let \( T, T^* \) be \( T \)-norms such that \( T^* \) dominates \( T \). If \( A_1 \) and \( A_2 \) be two \( T \)-fuzzy subalgebra of \( Y \), then the \( T^* \)-product of \( A_1 \) and \( A_2 \), \([A_1,A_2]_{T^*} \) is a \( T^* \)-fuzzy subalgebra of \( Y \). Since every epimorphic pre-image of an \( T \)-fuzzy subalgebra is a \( T \)-fuzzy subalgebra, the pre-images \( f^{-1}(A_1) \), \( f^{-1}(A_2) \) and \( f^{-1}([A_1,A_2]_{T^*}) \) are \( T \)-fuzzy subalgebras of \( X \). The next theorem provides the relation between \( f^{-1}([A_1,A_2]_{T^*}) \) and the \( T^* \)-product \([f^{-1}(A_1),f^{-1}(A_2)]_{T^*} \) of \( f^{-1}(A_1) \) and \( f^{-1}(A_2) \).

Theorem 4.2. Let \( f : X \to Y \) be an epimorphism of \( B \)-algebras. Let \( T, T^* \) be \( T \)-norms such that \( T^* \) dominates \( T \). Let \( A_1 \) and \( A_2 \) be two \( T \)-fuzzy subalgebra of \( Y \). If \([A_1,A_2]_{T^*} \) is the \( T^* \)-product of \( A_1 \) and \( A_2 \) and \([f^{-1}(A_1),f^{-1}(A_2)]_{T^*} \) is the \( T^* \)-product of \( f^{-1}(A_1) \) and \( f^{-1}(A_2) \), then
\[
f^{-1}([\alpha_{A_1},\alpha_{A_2}]_{T^*}) = [f^{-1}(\alpha_{A_1}),f^{-1}(\alpha_{A_2})]_{T^*}.
\]

Proof. For any \( x \in X \) we get,
\[
f^{-1}([\alpha_{A_1},\alpha_{A_2}]_{T^*})(x) = [\alpha_{A_1},\alpha_{A_2}]_{T^*}(f(x)) \\
= T^*(\alpha_{A_1}(f(x)),\alpha_{A_2}(f(x))) \\
= T^*(f^{-1}(\alpha_{A_1})f(x),f^{-1}(\alpha_{A_2})f(x)) \\
= [f^{-1}(\alpha_{A_1}),f^{-1}(\alpha_{A_2})]_{T^*}(x).
\]

\[\square\]
Lemma 4.1. [5], Let $T$ and $S$ be a t-norm. Then $T(T(x, y), T(z, t)) = T(T(x, z), T(y, t))$ for all $x, y, z$ and $t \in [0, 1]$

Theorem 4.3. Let $X = X_1 \times X_2$ be the direct product $B$-algebra of $B$-algebras $X_1$ and $X_2$. If $A_1 = \{< x, \alpha_{A_1}(x) > : x \in X \}$ and $A_2 = \{< x, \alpha_{A_2}(x) > : x \in X \}$ be two $T$-fuzzy subalgebra of $X_1$ and $X_2$ respectively, then $A = \{< x, \alpha_A(x) > : x \in X \}$ is a $T$-fuzzy subalgebra of $X$ defined by $\alpha_A = \alpha_{A_1} \times \alpha_{A_2}$ such that $\alpha_A(x_1, x_2) = (\alpha_{A_1} \times \alpha_{A_2})(x_1, x_2) = T(\alpha_{A_1}(x_1), \alpha_{A_2}(x_2))$ for all $(x_1, x_2) \in X_1 \times X_2$.

Proof. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any elements of $X$. Since $X$ is a $B$-algebra, we have,

$$\alpha_A(x \ast y) = \alpha_A((x_1, x_2) \ast (y_1, y_2)) = \alpha_A(x_1 \ast y_1, x_2 \ast y_2)$$

$$= (\alpha_{A_1} \times \alpha_{A_2})(x_1 \ast y_1, x_2 \ast y_2)$$

$$= T(\alpha_{A_1}(x_1 \ast y_1), \alpha_{A_2}(x_2 \ast y_2))$$

$$\geq T(T(\alpha_{A_1}(x_1), \alpha_{A_1}(y_1)), T(\alpha_{A_2}(x_2), \alpha_{A_2}(y_2)))$$

$$= T(T(\alpha_{A_1}(x_1), \alpha_{A_2}(x_2)), T(\alpha_{A_1}(y_1), \alpha_{A_2}(y_2)))$$

$$= T((\alpha_{A_1} \times \alpha_{A_2})(x_1, x_2), (\alpha_{A_1} \times \alpha_{A_2})(y_1, y_2))$$

$$= T(\alpha_A(x), \alpha_A(y)).$$

Hence, $A = \{< x, \alpha_A(x) > : x \in X \}$ is a $T$-fuzzy subalgebra of $X$.

5 Conclusions and Future Work

In this paper, $T$-fuzzy subalgebras of $B$-algebra are introduced and investigated some of their useful properties. Using imaginable property, imaginable $T$-fuzzy subalgebras of $B$-algebra has been constructed. Finally, the direct products and $T$-products of $T$-fuzzy subalgebras has been introduced and some important properties of it are also studied.

It is our hope that this work would other foundations for further study of the theory of $B$-algebras. In our future study of fuzzy structure of $B$-algebra, may be the following topics should be considered; (i) to find $T$-fuzzy closed ideals of $B$-algebras, (ii) to find interval-valued $T$-fuzzy subalgebras of $B$-algebra, (iii) to find interval-valued $T$-fuzzy closed ideals of $B$-algebra, (iv) to find intuitionistic $(T, S)$-fuzzy subalgebras of $B$-algebra, where $S$ is a given $t$-conorm.

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