Data Envelopment Analysis (DEA) with integer and negative inputs and outputs

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Abstract

Selecting an appropriate model in DEA to calculate the efficiency of DMUs has always been considered by science researchers. Original models of DEA have been implemented only for some technologies which have positive inputs and outputs feature. In this paper, first we examine semi-oriented radial measure (SORM) model to calculate the efficiency of units based on negative data. After providing a review of the disadvantages to the mentioned model, we present its modified model. Then a new model is offered to evaluate the main units in the presence of negative integer data. Finally, a numerical example is provided in which the efficiency of the units is calculated by using the mentioned models.

Keywords: Data Envelopment Analysis (DEA), negative data, integer input and output, SORM.

1 Introduction

Data Envelopment Analysis (DEA) [1] is a nonparametric method for measuring the efficiency of the decision making units (DMU) which was first introduced by Charnes, CooperandRhodesin (1978), [5] as the CCR model and then BCC model was introduced by Banker, Charnes and Cooper [4] to the realm of operations research and management science. Theoretically, The main assumption in all DEA models was that all input and output values are positive, but practically, we encounter many cases that violate this term and we ultimately have negative inputs and outputs. Among the proposed methods of dealing with negative data, the following models could be provided. SeifordandZhu [10] considered a positive and very small value of negative output. Another method was proposed by Halme et al. offering the measurement theory and deference of scale variables and the fraction in order to explain the reason for negative observations and also represented a reliable method for assessing interval scale units [8]. The other method which is pervasive is called RDM introduced by Portela et al. Modified slack-based measure model, called MSBM was represented by Sharp et al [12]. However, the latest method of behavior with negative data was provided by Emrouznejad et al [11]. which is based on SORM model and considered some variables which are both negative and positive for DMUs [6]. This model by using available variable changes was
not considered as a reliable method. Consequently, radial methods of DEA were modified for the evaluation of the efficiency of units by negative data. SORM model in input oriented was modified by KazemiMatin [9] and also Jahanshahloo et al. modified the output oriented. In addition, the evaluation of efficiency in order to supply a chain with negative data and also new models for the calculation of efficiency in decision making units with negative data was show by Jahanshahloo et al., [2], and the review of FDH model on negative data was provided [2]. In all DEA models that we faced, our assumption was that inputs and outputs of decision making units get negative value. In this paper, variables do not necessarily have integer term but the combination of variables does have integer term. It means that some variables are integer and some of them are real and the combination of all real variables is integer. So, it's not a classic issue but we should follow the right path to achieve it. Therefore, it's needed to run mathematical modeling. The main objective of this paper is to decide how to deal with decision making units that have negative and integer inputs and outputs and also represent a model whose scheme and pattern for each inefficiency unit in special term is true.

Section (2), we express SORM model.
Section (3), a new model for calculation of efficiency with negative and integer data is presented.
Section (4), numerical example is provided.
Section (5), conclusion is drawn.

2 SORM model

Decision making units (DMU) mean that by receiving an input vector such as \( X = (x_1, ..., x_m) \) an output vector like \( Y = (y_1, ..., y_n) \) is reproduced. Now, assume that "n" decision making units for evaluation exist. Then, by receiving an input vector like \( X_j = (x_{1j}, ..., x_{mj}) \), \( j \)-th of DMU produces an output vector like \( Y_j = (y_{1j}, ..., y_{nj}) \). In this case, it's assumed that input and output vector components are non-negative, if a variable gets negative value for some DMUs and also the others get some positive value. So, it’s an independent variable which can be written as the difference between two non-negative variables. In other words, we have:

\[
x = x_1 - x_2 x_3 \geq 0 , \quad x_2 \geq 0
\]

If \( x \geq 0 \) then \( x_1 = |x| , \quad x_2 = 0 \)
If \( x < 0 \) then \( x_2 = |x| , \quad x_1 = 0 \)

Emrouznejad et al. used this method to represent SORM model for the calculation of the efficiency of units which have negative data [6]. If a variable has positive value for some DMUs and gets negative value for the others, depending on the position of DMU and its positive or negative value, that variable can grow or decline in order to improve the main DMU's performance. If DMU has negative value in output position, it’s desirable to have a reduction in output value to increase output absolute value and to have an improvement in DMU performance. To solve this problem, based on the variable, having positive value for some DMUs and getting negative value for the others, it behaved as the sum of two variables. First, input and output indices are divided into the following category.

\[
I = \{ i \in \{ 1, ..., m \} : x_{ij} \geq 0, \ j = 1, ..., n \} \\
L = \{ l \in \{ 1, ..., m \} : \exists j \in \{ 1, ..., n \}; \ x_{lj} < 0 \} \\
R = \{ r \in \{ 1, ..., s \} : y_{rf} \geq 0, \ j = 1, ..., n \} \\
K = \{ k \in \{ 1, ..., s \} : \exists j \in \{ 1, ..., n \}; \ y_{kj} < 0 \} \\
I \cup L = \{ 1, ..., m \} , \ R \cup K = \{ 1, ..., s \} , I \cap L = \emptyset , R \cap K = \emptyset
\]

Assumed that input \( x_1 \) and output \( y_k \) have positive value for some DMUs and get negative value for the others. \( x_1^1, x_1^2, y_k^1, \) and \( y_k^2 \) would be defined for \( j \)-th of DMU as follows.
Note that we have $x_{ij}^2 \geq 0, x_{ij}^1 \geq 0$ while $x_{ij} = x_{ij}^1 - x_{ij}^2$ for all $j$.

Note that we have $y_{kj}^2 \geq 0, y_{kj}^1 \geq 0$ while $y_{kj} = y_{kj}^1 - y_{kj}^2$ for all $j$.

In the model presented by Emrouznejad et al. [6] based on radial method of DEA, we expect a decrease in inputs and an increase in DMU outputs. Our purpose in the input oriented is to minimize the input values $\text{DMU}_p$ which is under evaluation. According to the following term: $x_{ip} = x_{ip}^1 - x_{ip}^2$.

The find result: $\text{Min } x_{ip} = \text{Min } (x_{ip}^1 - x_{ip}^2) = \text{Min } x_{ip}^1 + \text{Max } x_{ip}^2$

So, the decrease $x_{ip}^1$ and the increase $x_{ip}^2$ are desired. Since, $x_{ip}^2$ behaves like an output; thus, we recognized it as a new output. In output oriented, our goal is to maximize output values $\text{DMU}_p$ which are under evaluation.

Accordingly: $y_{kp} = y_{kp}^1 - y_{kp}^2$ Leads to:

$\text{Max } y_{kp} = \text{Max } (y_{kp}^1 - y_{kp}^2) = \text{Max } y_{kp}^1 + \text{Min } y_{kp}^2$

As a result the increase $y_{kp}^1$ and the decrease $y_{kp}^2$ are desired. Since, $y_{kp}^2$ behaves like an input; we consider it as a new input. So, the SORM model proposed by Emrouznejad et al., [6] is like model (1).

\[
\begin{align*}
\text{Min } & \quad h \\
\text{s.t.} & \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq hx_{ip} \quad i \in I \quad (a) \\
& \quad \sum_{j=1}^{n} \lambda_j x_{ij}^1 \leq hx_{ip}^1 \quad l \in L \quad (b) \\
& \quad \sum_{j=1}^{n} \lambda_j x_{ij}^2 \geq hx_{ip}^2 \quad l \in L \quad (c) \\
& \quad \sum_{j=1}^{n} \lambda_j y_{kj} \geq y_{rp} \quad r \in R \quad (d) \\
& \quad \sum_{j=1}^{n} \lambda_j y_{kj}^1 \geq y_{kp}^1 \quad k \in K \quad (e) \\
& \quad \sum_{j=1}^{n} \lambda_j y_{kj}^2 \leq y_{kp}^2 \quad k \in K \quad (f) \\
& \quad \sum_{j=1}^{n} \lambda_j = 1 \\
& \quad \lambda_j \geq 0 \quad j = 1, \ldots, n
\end{align*}
\]

(1)

In fact, Emrouznejad et al., [6], in SORM model for input $x_{ip}$, introduced two constraints. By multiplying both sides of constraint (c) by $-1$ and the summation with the constraint (b), we have: $\sum_{j=1}^{n} \lambda_j (x_{ij}^1 - x_{ij}^2) \leq h(x_{ip}^1 - x_{ip}^2)$ and since

$x_{ip} = x_{ip}^1 - x_{ip}^2, x_{ij} = x_{ij}^1 - x_{ij}^2$ we have created the initial model as obtained before disaggregating $x_i$. For output, $y_k$, there exists the same reason. But the main problem we face, is that in following equations the right side of an equation is to reach zero in order to prove that it's true and also their result from multiplying is zero.

\[
\begin{align*}
\sum_{j=1}^{n} \lambda_j x_{ij}^1 & = hx_{ip}^1 - s_i^- \quad l \in L \\
\sum_{j=1}^{n} \lambda_j x_{ij}^2 & = hx_{ip}^2 + s_i^+ \quad l \in L
\end{align*}
\]
In this case, if our DMU which is under evaluation, is a DMU whose input vector has a negative value, then $x_{ip}^1 = 0$ and $\sum_{j=1}^{n} \lambda_j x_{ij}^1 + s_i^- = 0 \quad l \in L$

Summation of infinite numbers of negative data became zero, so each of them must be zero as well.

Consequently, $s_i^- = 0$ and $\sum_{j=1}^{n} \lambda_j x_{ij}^1 = h x_{ip}^1 - s_i^- = 0 \quad l \in L$

In this case, the above-mentioned method is right. But if our DMU which is under evaluation, is a DMU whose input variable has a positive value, we could not show that in one of two equations, the value on the right side will necessarily become zero. Hence, the difference may become positive and finally violate this method. SORM model in output oriented presented by Emrouznejad et al, [6] is like model (2).

$$\begin{align*}
\text{Max} & \quad h \\
\text{s.t.} & \quad \sum_{j=1}^{n} \lambda_j x_{ij}^1 \leq x_{ip}^1 \quad i \in I \\
& \quad \sum_{j=1}^{n} \lambda_j x_{ij}^2 \geq x_{ip}^2 \quad l \in L \\
& \quad \sum_{j=1}^{n} \lambda_j y_{kj}^1 \geq h y_{kp}^1 \quad k \in K \\
& \quad \sum_{j=1}^{n} \lambda_j y_{kj}^2 \leq h y_{kp}^2 \quad k \in K \\
& \quad \sum_{j=1}^{n} \lambda_j = 1 \\
& \quad \lambda_j \geq 0 \quad j = 1, ..., n
\end{align*}$$

(2)

SORM model has a defect which was resolved by KazemiMatin[9] and a new model was provided that increases the efficiency. We explained that in input oriented, our goal is to minimize the input value DMU, which is under evaluation. According to $x_{ip} = x_{ip}^1 - x_{ip}^2$. We conclude that

$$\begin{align*}
\text{Min} & \quad x_{ip} = \text{Min} (x_{ip}^1 - x_{ip}^2) = \text{Min} x_{ip}^1 + \text{Max} x_{ip}^2
\end{align*}$$

So, the decrease value $x_{ip}^1$ and the increase value $x_{ip}^2$ are desirable. Since, $x_{ip}^2$ behaves like an output; we consider it as a new output.

According to constraint $\sum_{j=1}^{n} \lambda_j x_{ij}^2 \geq h x_{ip}^2 \quad l \in L$ and by minimizing "h" in input oriented the ideal result was not achieved. There's exactly the same problem for SORM model in output oriented, and in this case, our purpose is to maximize the output values DMU, which are under evaluation.

According to $y_{kp} = y_{kp}^1 - y_{kp}^2$.

We conclude that: $\text{Max} y_{kp} = \text{Max} (y_{kp}^1 - y_{kp}^2) = \text{Max} y_{kp}^1 + \text{Min} y_{kp}^2$

So, the increase value $y_{kp}^1$ and the decrease value $y_{kp}^2$ are desirable. Since, $y_{kp}^2$ behaves like an input; we consider it as a new input. According to the constraint $\sum_{j=1}^{n} \lambda_j y_{kj}^2 \leq h y_{kp}^2 \quad k \in K$

and by maximizing "h" in output oriented, the ideal result was not achieved. SORM model in input oriented, introduced by KazemiMatin [9] is model (3).
In model (3), if \( h=0 \), then DMU which is under evaluation, is efficient. Otherwise, it is non-efficient. In fact, KazemiMatin [9] using represented technique in the combination of the DEA models and introducing the vector \( d = (−x_{ip}, −\ell^1_{ip}, \ell^2_{ip}, 0, 0, 0) \) achieved the desirable result. Since model (3) doesn’t calculate the efficiency, by implementing the change rate of the variable \( h \), we achieve model (4) which calculates the efficiency of the decision making unit.

\[
\begin{align*}
\text{Min} & \quad h' \\
\text{s.t} & \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq h' x_{ip} \quad i \in I \\
& \quad \sum_{j=1}^{n} \lambda_j x_{ij}^1 \leq h' \ell^1_{ip} \quad l \in L \\
& \quad \sum_{j=1}^{n} \lambda_j x_{ij}^2 \geq (2 − h') x_{ip}^2 \quad l \in L \\
& \quad \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp} \quad r \in R \\
& \quad \sum_{j=1}^{n} \lambda_j y_{kj}^1 \geq y_{kp}^1 \quad k \in K \\
& \quad \sum_{j=1}^{n} \lambda_j y_{kj}^2 \leq y_{kp}^2 \quad k \in K \\
& \quad \sum_{j=1}^{n} \lambda_j = 1 \\
& \quad \lambda_j \geq 0 \quad j=1, ..., n
\end{align*}
\]

In SORM model in the output oriented, the same problem is noticed with SORM model in the input oriented. Therefore, model (5) was proposed by Jahanshahloo et al. [2] to determine the efficiency too.

\[
\begin{align*}
\text{Max} & \quad h \\
\text{s.t} & \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{ip} \quad i \in I \\
& \quad \sum_{j=1}^{n} \lambda_j x_{ij}^1 \leq x_{ip}^1 \quad l \in L \\
& \quad \sum_{j=1}^{n} \lambda_j x_{ij}^2 \geq x_{ip}^2 \quad l \in L \\
& \quad \sum_{j=1}^{n} \lambda_j y_{rj} \geq h y_{rp} \quad r \in R \\
& \quad \sum_{j=1}^{n} \lambda_j y_{kj}^1 \geq h y_{kp}^1 \quad k \in K \\
& \quad \sum_{j=1}^{n} \lambda_j y_{kj}^2 \leq \frac{1}{h} y_{kp}^2 \quad k \in K \\
& \quad \sum_{j=1}^{n} \lambda_j = 1 \\
& \quad \lambda_j \geq 0 \quad j = 1, ..., n
\end{align*}
\]

3 SORM model in input oriented with positive and negative data

In this section, we develop a new model that has a variable which receives positive value for some DMUs and gets negative value for some others. Furthermore, inputs and outputs should be true in integer term. For instance, in case net profit of a company (\$ currency) is considered as an output variable and company faces losses. Let us assume that DMU is under evaluation and typically its primary output \( y_{1p} \) is
negative and also we know that our purpose is to increase outputs. It's written as a difference of two non-negative numbers and also we have

\[ \text{Max } y_{1p} = \text{Max}(y_{1p}^1 - y_{1p}^2) = \text{Max} y_{1p}^1 + M in y_{1p}^2 \]

Hence, the increase rate \( y_{1p}^1 \) and the decrease rate \( y_{1p}^2 \) rate are desirable. The main issue is to improve \( y_{1p} \) that could be minus of two integer or non-integer numbers, but the improved \( y_{1p} \) should be integer.

**Example 3.1:** let us assume that when DMU \( p \) is under evaluation its primary output is \( y_{1p} = -20 \). We have \( y_{1p} = -20 = 0 - 20 \).

The increase of zero and the decrease of 20 are aimed. For instance, zero increases to 5 and 20 decreases to 15. We have \( y_{1p} = 5 - 15 = -10 \). In consequence, output value improves. The important value is \( y_{1p} \) that must be integer and negative. In other words, we can write: \( y_{1p} = 5.4 - 15.4 = -10 \).

So, we must assure that \( y_{1p}^1, y_{1p}^2 \) could be any number and just their difference is important and hereby the constraint \( z_k = (y_{kp}^1 + s_{k}^2) - (y_{kp}^2 - s_{k}^2) \) will be added and it is not needed to work under the condition that \( y_{1p}^1, y_{1p}^2 \) must be integer. Because under the mentioned condition, all states of difference of two non-integer numbers are ignored. Exactly for the same reason, the constraint

\[ z_l = (h^l x_{lp}^1 - s_l^l) - ((2 - h^l) x_{lp}^2 + s_l^l) \]

was added to the model.

\[ \text{Min } h^l \]

\[ \begin{align*}
\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- &= h^l x_{lp}^1 & & i \in L \\
\sum_{j=1}^{n} \lambda_j x_{ij}^1 + s_i^- &= h^l x_{lp}^1 & & l \in L \\
\sum_{j=1}^{n} \lambda_j x_{ij}^2 - s_i^+ &= (2 - h^l) x_{lp}^2 + s_i^+ & & l \in L \\
\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^l &= y_{lp} & & r \in R \\
\sum_{j=1}^{n} \lambda_j y_{rj}^1 - s_r^l &= y_{lp}^1 & & k \in K \\
\sum_{j=1}^{n} \lambda_j y_{rj}^2 + s_r^l &= y_{lp}^2 & & k \in K \\
\sum_{j=1}^{n} \lambda_j (y_{kp}^1 + s_r^2) - (y_{kp}^2 - s_r^2) & & k \in K \\
\sum_{j=1}^{n} \lambda_j &= 1 \\
\lambda_j &\geq 0 & & j = 1, ..., n \\
z_l, z_k &\in \mathbb{Z} & & l \in L, k \in K \\
s_{i}, s_{i}^l, s_{i}^+, s_{r}^2, s_{r}^+ &\geq 0 & & i \in I, l \in L, k \in K, r \in R
\end{align*} \]

**Example 3.2:** assume there are 4 DMUs with following input and output:

DMU \( A \) = \((-3,1), \) DMU \( B \) = \((-1,3), \) DMU \( C \) = \((2,4), \) DMU \( D \) = \((4,3) \)

Output of all DMUs is positive but input is positive for some of them and negative for the others. As we expressed:

\[ \text{DMU } A = (0,3,1), \text{DMU } B = (0,1,3), \text{DMU } C = (2,0,4), \text{DMU } D = (4,0,3) \]

In other words, primary feature which was shown the inputs, turned into two features:

\[ x = x_1 - x_2; \quad x_1 \geq 0, x_2 \geq 0. \]

In fact by this procedure, one dimension will be added.

**Theorem 3.1:** In model (6) we have: \( 0 < h'' \leq 1 \).

**Proof:** First we show that, \( h'' \leq 1 \).

Supposing that the slack variables are zero, and considering: \( \lambda_j = 0, \lambda_p = 1 \forall j \neq p \)

then we have \( h'' = 1 \). This is a feasible solution for model (6) and as the objective function is a minimizing one: \( h'' \leq 1 \).
Now, we are to prove that, \( h'^* \neq 0 \). If we suppose the contrary of this term, then \( h'^* = 0 \) and the first constraint is considered unequal.

Then considering that for each \( j \) we have \( \lambda_j \geq 0 \) and, on the other hand we have for \( \forall i \in I \) \( x_{ij} \geq 0 \). Hence, based on the mentioned assumption for the first constraint in model (6), we have \( \sum_{j=1}^{n} \lambda_j x_{ij} \leq 0 \). This equation could not exist.

\( h'^* \) also could not be negative because in this case, in the first constraint on the right side of the unequal, there’s a negative number and on the left side for total finite number, there’s a positive number.

**Theorem 3.2:** In all optimal solution of the model (6), \( \lambda'^* \neq 0 \).

**Proof:** By contradiction we assume that there is an optimal solution in which, \( \lambda'^* = 0 \). We consider a condition in which the DMU under evaluation, has a positive output, it means: \( y_{rp} > 0 \) \( r \in R \).

We consider the following constraint without the slack variable:

\[
\sum_{j=1}^{n} \lambda_j y_{rj} - s^+_r = y_{rp} \quad r \in R.
\]

Therefore we have \( \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp} \quad r \in R \).

Now, if \( \lambda'^*_j = 0 \).

We have: \( y_{rp} \leq 0 \) \( r \in R \).

As we supposed from the first step that: \( y_{rp} > 0 \) \( r \in R \).

Consequently, the contradiction assumption is invalid and the sentence is true for the DMU under evaluation which has a positive output. But if DMU under evaluation has not a positive output, in other words \( y_{kp}^2 < 0 \) \( k \in K \).

Then we consider the following constraint without the slack variable.

\[
\sum_{j=1}^{n} \lambda_j y_{kj}^2 + s_k = y_{kp}^2 \quad k \in K.
\]

Thus we have: \( \sum_{j=1}^{n} \lambda_j y_{kj}^2 \leq y_{kp}^2 \) \( k \in K \).

Now, if \( \lambda'^*_j = 0 \).

we have \( y_{kp}^2 \geq 0 \) \( k \in K \).

but we have supposed from the first step that: \( y_{kp}^2 < 0 \) \( k \in K \).

Therefore, the contradiction assumption is invalid and the sentence is true while DMU which is under evaluation has not a positive output.

**Theorem 3.3:** DMU \( p \) in "T", is strongly efficient, if and only if in every optimal solution of model (6) we have:

- a) \( h'^* = 1 \)
- b) \( (s'^-*, s'^*) = 0 \)

**Proof:** first assume that DMU \( p \) in "T" is strongly efficient and we show that sentences "a" and "b" are true. By contradiction we assume that at least one of 2 terms "a" or "b" is not true. First state: "a" term is not true.

It means that \( h'^* \neq 1 \). Suppose that \((\lambda'^*, s'^-, s'^+, h'^*)\) is the optimal solution for model (6) that \( h'^* < 1 \). We write the equations for the optimal solution as follows:

\[
\sum_{j=1}^{n} A_j \ x_{ij} + s_{i}^- = h'^* \ x_{ip} \leq x_{ip} \quad i \in I
\]

\[
\sum_{j=1}^{n} A_j \ x_{ij}^2 + s_{i}^+ = h'^* \ x_{ip}^2 < x_{ip}^2 \quad l \in L
\]

\[
\sum_{j=1}^{n} A_j \ y_{rj} - s_{r}^+ = y_{rp} \quad r \in R
\]

\[
\sum_{j=1}^{n} A_j \ y_{kj}^2 - s_{k}^- = y_{kp}^2 \quad k \in K
\]

\[
\sum_{j=1}^{n} A_j \ y_{kp}^2 + s_{k}^+ = y_{kp}^2 \quad k \in K
\]
The reason that we do not make strict unequal equations for primary constraint is that $x_{ip}$ may have a zero component; so $h^* x_{ip}$ has zero components. Therefore $x_{ip}$ and $h^* x_{ip}$ are equal in one component. Two other constraints based on the definition of "L" are strict and also the reason that in the third constraint, the bigger sign is converted to the small sign is that $h^*$ is a positive number and less than one, so $(2-h^*)$ is a positive number and bigger than one and less than two, which by multiplying it by $x_{ip}^2$ which is a negative number; it is less than $x_{ip}$. Notice that considering DMU$_p$ under evaluation and its inputs or outputs getting negative or positive value, equations for the optimal solution would be different. It's important that represented proof is true for all cases. As such we have,

$$A = \left( \begin{array}{c} -\sum_{j=1}^{n} \lambda_j^* x_{ij} \\ -\sum_{j=1}^{n} \lambda_j^* x_{ij}^2 \\ \sum_{j=1}^{n} \lambda_j^* y_{rj} \\ \sum_{j=1}^{n} \lambda_j^* y_{kj}^2 \\ -\sum_{j=1}^{n} \lambda_j^* y_{kj} \\ \end{array} \right) \preceq \left( \begin{array}{c} -x_{ip} \\ -x_{ip}^2 \\ y_{rp} \\ y_{kp}^2 \\ y_{kp} \\ \end{array} \right)$$

We found a point such as "A" in "T" that is better than DMU$_p$ while we suppose that DMU$_p$ is strongly efficient in "T" and consequently, the contradiction assumption is invalid and the sentence is true.

Second state: $h^* = 1$ and $(s^{++}, s^{--}) \neq 0$.

In other words, at least one of the slack variables is positive. Then we have:

$$\sum_{j=1}^{n} \lambda_j^* x_{ij} + s_i^{--} = x_{ip}, \quad i \in I$$
$$\sum_{j=1}^{n} \lambda_j^* x_{ij}^2 = x_{ip}^2, \quad l \in L$$
$$\sum_{j=1}^{n} \lambda_j^* y_{rj} - s_r^{--} = y_{rp}, \quad r \in R$$
$$\sum_{j=1}^{n} \lambda_j^* y_{kj} - s_k^{--} = y_{kp}, \quad k \in K$$

Since $(s^{++}, s^{--}) \neq 0$, then we can write:

$$A = \left( \begin{array}{c} -\sum_{j=1}^{n} \lambda_j^* x_{ij} \\ -\sum_{j=1}^{n} \lambda_j^* x_{ij}^2 \\ \sum_{j=1}^{n} \lambda_j^* y_{rj} \\ \sum_{j=1}^{n} \lambda_j^* y_{kj}^2 \\ \sum_{j=1}^{n} \lambda_j^* y_{kj} \\ \end{array} \right) \preceq \left( \begin{array}{c} -x_{ip} \\ -x_{ip}^2 \\ y_{rp} \\ y_{kp}^2 \\ y_{kp} \\ \end{array} \right)$$

and it implies that we found one point such as "A" in "T" which is better than DMU$_p$ while we suppose that DMU$_p$ is strongly efficient in "T". The contradiction assumption is invalid and the sentence is true.

By contradiction we assume that: $h^* = 1$ and $(s^{++}, s^{--}) = 0$. Then we show that DMU$_p$ is strongly efficient in "T". By contradiction we assume: the term DMU$_p$ which is strongly efficient is not in "T". 

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Consequently, there is a better point such as \((X', Y')\) in the production possibility set. It means that we have

\[
\exists (X', Y') \in T, \exists \lambda' \geq 0 : A = \begin{pmatrix}
- \sum_{j=1}^{n} \lambda_j' x_{ij} \\
- \sum_{j=1}^{n} \lambda_j' x_{ij}^1 \\
\sum_{j=1}^{n} \lambda_j' x_{ij}^2 \\
\sum_{j=1}^{n} \lambda_j' y_{rj} \\
\sum_{j=1}^{n} \lambda_j' y_{kj}^1 \\
- \sum_{j=1}^{n} \lambda_j' y_{kj}^2
\end{pmatrix} \neq \begin{pmatrix}
-x_{ip} \\
-x_{ip}^1 \\
x_{ip}^2 \\
y_{rp} \\
y_{kp}^1 \\
y_{kp}^2
\end{pmatrix}
\]

At least one of the constraints of the mentioned equation is unequally strict. So we have:

\[
\exists t : \sum_{j=1}^{n} \lambda_j' x_{ij} < x_{tp} \quad \text{or} \quad \sum_{j=1}^{n} \lambda_j' x_{ij}^1 < x_{tp}^1 \\
\exists t' : \sum_{j=1}^{n} \lambda_j' x_{ij} > x_{tp}^2 \\
\exists t'' : \sum_{j=1}^{n} \lambda_j' x_{ij}^2 > x_{tp}^{2'} \\
\exists d : \sum_{j=1}^{n} \lambda_j' y_{dj} > y_{dp} \quad \text{or} \\
\exists d' : \sum_{j=1}^{n} \lambda_j' y_{dj}^1 > y_{dp}^1 \\
\exists d'' : \sum_{j=1}^{n} \lambda_j' y_{dj}^2 < y_{dp}^{2'} \tag{7}
\]

Now, we convert the non-equations to the equations:

\[
\sum_{j=1}^{n} \lambda_j' x_{ij} + s_i^- = x_{ip} \quad i \in I \\
\sum_{j=1}^{n} \lambda_j' x_{ij}^1 + s_i^- = x_{ip}^1 \quad l \in L \\
\sum_{j=1}^{n} \lambda_j' x_{ij}^2 - s_i^+ = x_{ip}^2 \quad l \in L \\
\sum_{j=1}^{n} \lambda_j' y_{rj} - s_r^+ = y_{rp} \quad r \in R \\
\sum_{j=1}^{n} \lambda_j' y_{kj}^1 - s_k^- = y_{kp}^1 \quad k \in K \\
\sum_{j=1}^{n} \lambda_j' y_{kj}^2 + s_k^- = y_{kp}^2 \quad k \in K
\]

On condition that all slack variables are not zero, because if slack variables are zero, we achieve equal condition, while in equation (7) we have unequal sign. In the objective function from model (6) we have Min \(h'\), according to our assumption, summation of slack variables is zero. So, if we write the row of objective function in the following form, there’s no change in the value of the optimal solution and also, \(Z^* = 1\) and Min \(Z = h' - \varepsilon (\sum_{i \in I} s_i^- + \sum_{i \in I} (s_i^- + s_i^+) + \sum_{r \in R} s_r^+ + \sum_{k \in K} (s_k^- + s_k^+))\)

On the other hand, because \((\lambda', s'^-, s'^+, h') = 1\) a feasible solution with total slack variables is zero. we have

\[
Z = h' - \varepsilon (\sum_{i \in I} s_i^- + \sum_{i \in I} (s_i^- + s_i^+) + \sum_{r \in R} s_r^+ + \sum_{k \in K} (s_k^- + s_k^+)) < 1.
\]

Because \(\varepsilon > 0\) and \(\varepsilon (\sum_{i \in I} s_i^- + \sum_{i \in I} (s_i^- + s_i^+) + \sum_{r \in R} s_r^+ + \sum_{k \in K} (s_k^- + s_k^+)) \neq 0\)

Therefore \(Z < Z^*\), we found a feasible solution whose amount for objective function is less than the optimal amount. The contradiction assumption is invalid and the sentence is true. Of course, it’s possible that DMU which is under evaluation does not have negative or positive input or output, but in any case, the represented proof is right.
Theorem 3.4: If $h^*$ is the optimal solution for SORM model in input oriented, with positive and negative data, then \((h^* x_{ip}, h^* x_{ip}^1, (2-h^*) x_{ip}^2, y_{rp}, y_{kp}^1, y_{kp}^2) \in \partial T\).

Proof: it is to show that if this point is evaluated again, then \((h^*)' = 1\).

By contradiction we assume that this point is assessed and we find $h' < 1$ In other words, we have:

\[
\begin{align*}
\text{Min} & \quad h \\
\text{s.t} & \quad \sum_{j=1}^{n} \lambda_j x_{ij} + s_{i}^- = h(h^* x_{ip}) & i \in I \\
& \quad \sum_{j=1}^{n} \lambda_j x_{ij} + s_{i}^- = h(h^* x_{ip}) & l \in L \\
& \quad \sum_{j=1}^{n} \lambda_j x_{ij}^2 - s_{i}^+ = (2-h)((2-h^*) x_{ip}) & l \in L \\
& \quad z_i = \left(h(h^* x_{ip}^1) - s_{i}^-\right) - \left((2-h)((2-h^*) x_{ip}) + s_{i}^+\right) \\
& \quad \sum_{j=1}^{n} \lambda_j y_{rj} - s_{i}^+ = y_{rp} & r \in R \\
& \quad \sum_{j=1}^{n} \lambda_j y_{kj}^1 - s_{i}^k = y_{kp}^1 & k \in K \\
& \quad \sum_{j=1}^{n} \lambda_j y_{kj}^2 + s_{i}^k = y_{kp}^2 & k \in K \\
& \quad z_k = (y_{kp}^1 + s_{i}^+) - (y_{kp}^2 - s_{i}^-) \\
& \quad \sum_{j=1}^{n} \lambda_j = 1 & (8) \\
\end{align*}
\]

Suppose that the optimal solution for model (8) is \((\lambda', s'^-, s'^+, h')\) which $h' < 1$.

\[
\begin{align*}
\text{Min} & \quad h' \\
\text{s.t} & \quad \sum_{j=1}^{n} \lambda'_j x_{ij} + s'^-_i = (h'^* x_{ip}) & i \in I \\
& \quad \sum_{j=1}^{n} \lambda'_j x_{ij}^1 + s'^-_i = (h'^* x_{ip}) & l \in L \\
& \quad \sum_{j=1}^{n} \lambda'_j x_{ij}^2 - s'^+_i = ((2-h')(2-h^*) x_{ip}) & l \in L \\
& \quad z_i = \left((h'h^* x_{ip}^1) - s'^-_i\right) - \left(((2-h')(2-h^*) x_{ip}) + s'^+_i\right) \\
& \quad \sum_{j=1}^{n} \lambda'_j y_{rj} - s'^+_i = y_{rp} & r \in R \\
& \quad \sum_{j=1}^{n} \lambda'_j y_{k}^1 - s'^+_i = y_{kp}^1 & k \in K \\
& \quad \sum_{j=1}^{n} \lambda'_j y_{k}^2 + s'^+_i = y_{kp}^2 & k \in K \\
& \quad z_k = (y_{kp}^1 + s'^+_i) - (y_{kp}^2 - s'^-_i) \\
& \quad \sum_{j=1}^{n} \lambda'_j = 1 & (9) \\
\end{align*}
\]

In this case \((\lambda', s'^-, s'^+, h'^*)\) it is feasible that $h'h' < h'$. So, unlike being optimal of $h'$. The contradiction assumption is invalid and the sentence is true.

Theorem 3.5: the achieved Point of picture from SORM model with integer and negative data in input oriented which is in the following form, is efficient.
A = \begin{pmatrix}
\sum^{n}_{j=1} \lambda_{ij} x_{ij} \\
\sum^{n}_{j=1} \lambda_{ij} x_{ij}^+ \\
\sum^{n}_{j=1} \lambda_{ij} x_{ij}^- \\
\sum^{n}_{j=1} \lambda_{ij} y_{rj} \\
\sum^{n}_{j=1} \lambda_{ij} y_{kj} \\
\sum^{n}_{j=1} \lambda_{ij} y_{kj}^+ \\
\sum^{n}_{j=1} \lambda_{ij} y_{kj}^- \\
\sum^{n}_{j=1} \lambda_{ij} y_{kj}^+
\end{pmatrix}
= \begin{pmatrix}
h x_{ip} - s_i^- \\
h x_{ip} - s_i^+ \\
(2-h) x_{ip}^+ + s_i^+ \\
y_{rp} + s_r^+ \\
y_{kp}^+ + s_k^+ \\
y_{kp}^- - s_k^-
\end{pmatrix}

Proof: Suppose that in the evaluation of DMU by SORM model with integer and negative data in input oriented, we achieve the optimal solution \((\lambda^*, s^-, s^+, h^*)\). We are to show that,

\begin{align*}
A = \begin{pmatrix}
h x_{ip} - s_i^- \\
h x_{ip} - s_i^+ \\
(2-h) x_{ip}^+ + s_i^+ \\
y_{rp} + s_r^+ \\
y_{kp}^+ + s_k^+ \\
y_{kp}^- - s_k^-
\end{pmatrix}
\end{align*}
in "T" is strongly efficient. or in other words, in the evaluation of "A" point, \(h = 1\) and slack variables are zero. By solving SORM model with integer and negative data in input oriented for the evaluation of point "A", we achieved the optimal solution \((\lambda', s^-, s^+, h')\), it is to show that \(h' = 1\) and slack variables are zero. By putting the optimal solution \((\lambda', s^-, s^+, h')\) in model (6) and also by putting the components of the point A, which is under evaluation, we achieve model (10).

Min \(h\)

\(s.t\)

\begin{align*}
\sum^{n}_{j=1} \lambda_{ij}^' x_{ij} + s_i^- &= h' (h' x_{ip} - s_i^-) \quad i \in I \\
\sum^{n}_{j=1} \lambda_{ij}^' x_{ij}^+ + s_i^+ &= h' (h' x_{ip} - s_i^-) \quad l \in L \\
\sum^{n}_{j=1} \lambda_{ij}^' x_{ij}^+ - s_i^- &= (2-h') ((2-h') x_{ip}^+ + s_i^+) \quad l \in L \\
z_l &= (h' (h' x_{ip} - s_i^-) - s_i^-) - ((2-h') (2-h') x_{ip}^+ + s_i^+) + s_i^+ \\
\sum^{n}_{j=1} \lambda_{ij}^' y_{rj} - s_r^+ &= y_{rp} + s_r^+ \quad r \in R \\
\sum^{n}_{j=1} \lambda_{ij}^' y_{kj}^+ - s_k^+ &= y_{kp}^+ + s_k^+ \quad k \in K \\
\sum^{n}_{j=1} \lambda_{ij}^' y_{kj}^- + s_k^- &= y_{kp}^- - s_k^- \quad k \in K \\
z_k &= ((y_{kp}^+ + s_k^+) + s_k^+) - ((y_{kp}^- - s_k^-) - s_k^-) \\
\sum^{n}_{j=1} \lambda_{ij}^' &= 1 \quad (10)
\end{align*}

\(\lambda_{ij}^' \geq 0\)

\(z_l, z_k \in \mathbb{R}\)

\(s_i^-, s_i^+, s_l^-, s_l^+, s_r^+, s_k^+, s_k^- \geq 0\)

By doing simple mathematical operations and taking the slack variables from the left side and putting them on the right side, we have:
Without making any damage to the proof, we suppose \( \lambda_j = \lambda_j \) then \((\lambda, \overline{s}^-, \overline{s}^+, \overline{h})\) is a feasible answer for our problem. Well, so far we have found two solutions for the problem one of which is feasible and the other one is optimal. If \( h' < 1 \) then we have \( \overline{h} = h' \) then \( \overline{h} < h' \) which is opposed to being optimal of \( h' \).

So it is to have \( h' = 1 \). Now, it must be show that the summation of slack variables is zero. In other words, it is to prove that

\[
\sum_{l \in L} s^{-l}_i + \sum_{l \in L} s^{-l}_i + \sum_{l \in L} s^{+l}_i + \sum_{r \in R} s^{+r}_i + \sum_{k \in K} s^{+k}_i + \sum_{k \in K} s^{-k}_i = 0
\]

Let us suppose that \( B_2 \) is the optimal solution of phase (\( \Pi \)) in the evaluation of the picture point in the two-phase problem.

\[
B_2 = \sum_{l \in L} s^{-l}_i + \sum_{l \in L} s^{-l}_i + \sum_{l \in L} s^{+l}_i + \sum_{r \in R} s^{+r}_i + \sum_{k \in K} s^{+k}_i + \sum_{k \in K} s^{-k}_i
\]

By putting the values of the slack variables in equation \( B_1 \) from equations of model (8), having \( h' = 1 \) and solving simple mathematical equations, we have:

\[
B_1 = \sum_{l \in L} s^{-l}_i + \sum_{l \in L} s^{-l}_i + \sum_{l \in L} s^{+l}_i + \sum_{r \in R} s^{+r}_i + \sum_{k \in K} s^{+k}_i + \sum_{k \in K} s^{-k}_i
\]

\[
= \sum_{i \in I}(h's^{-i} + s^{+i}) + \sum_{i \in I}(h's^{-i} + s^{+i}) + \sum_{i \in I}(2 - h's^{+i} + s^{-i}) + \sum_{r \in R}(s^{+r}_i + s^{+r}_i) + \sum_{r \in R}(s^{+r}_i + s^{+r}_i) + \sum_{r \in R}(s^{+r}_i + s^{+r}_i) + \sum_{r \in R}(s^{+r}_i + s^{+r}_i) + \sum_{r \in R}(s^{+r}_i + s^{+r}_i) + \sum_{k \in K}(s^{+k}_i + s^{+k}_i) + \sum_{k \in K}(s^{-k}_i + s^{-k}_i)
\]

\[
= \sum_{i \in I}(s^{-i}_i) + \sum_{i \in I}(s^{+i}_i) + \sum_{i \in I}(s^{+i}_i) + \sum_{k \in K}(s^{+k}_i) + \sum_{k \in K}(s^{-k}_i)
\]

\[
B_2 = \sum_{i \in I}(s^{-i}_i) + \sum_{i \in I}(s^{+i}_i) + \sum_{i \in I}(s^{+i}_i) + \sum_{k \in K}(s^{+k}_i) + \sum_{k \in K}(s^{-k}_i)
\]

\[
B_3 = \sum_{i \in I}(s^{-i}_i) + \sum_{i \in I}(s^{+i}_i) + \sum_{i \in I}(s^{+i}_i) + \sum_{k \in K}(s^{+k}_i) + \sum_{k \in K}(s^{-k}_i)
\]

then \( B_1 = B_2 + B_3 \)

And it should be shown that \( B_3 = 0 \). By contradiction we assume that \( B_3 > 0 \), hence \( B_2 < B_1 \), and it is opposed to the fact that \( B_2 \) is the optimal solution of phase (\( \Pi \)) in the evaluation of the picture point. Consequently the contradiction assumption is invalid and \( B_3 = 0 \) and in consequence, the summation of
slack variables became zero. Because \( h' = 1 \), the picture point is strongly efficient. SORM model in output oriented with integer and negative data is in the following form.

\[
\begin{align*}
\text{Max} & \quad h \\
\text{s.t} & \quad \sum_{j=1}^{n} \lambda_j x_{ij} + s_{i}^{-} = x_{ip} \quad i \in I \\
& \quad \sum_{j=1}^{n} \lambda_j x_{ij} + s_{i}^{-} = x_{ip} \quad l \in L \\
& \quad \sum_{j=1}^{n} \lambda_j x_{ij} - s_{i}^{+} = x_{ip} \quad l \in L \\
& \quad z_{i} = \left( x_{ip}^{1} - s_{i}^{-}\right) - \left( x_{ip}^{2} + s_{i}^{+}\right) \\
& \quad \sum_{j=1}^{n} \lambda_j y_{rj} - s_{r}^{+} = h y_{rp} \quad r \in R \\
& \quad \sum_{j=1}^{n} \lambda_j y_{jk} - s_{k}^{+} = h y_{kp} \quad k \in K \\
& \quad \sum_{j=1}^{n} \lambda_j y_{jk} + s_{k}^{-} = \frac{1}{h} y_{kp} \quad k \in K \\
& \quad z_{k} = \left(h y_{kp}^{1} + s_{k}^{-}\right) - \left(\frac{1}{h} y_{kp}^{2} - s_{k}^{+}\right) \\
& \quad \sum_{j=1}^{n} \lambda_j = 1 \\
& \quad \lambda_{j} \geq 0 \quad j = 1, \ldots, n \\
& \quad z_{i} , z_{k} \in \mathbb{R} \quad l \in L, k \in K \\
& \quad s_{i}^{-}, s_{i}^{+}, s_{k}^{-}, s_{k}^{+}, s_{k}^{-} \geq 0 \quad i \in I, l \in L, k \in K, r \in R
\end{align*}
\]

**Theorem 3.6:** In model (11) we have \( h^* \geq 1 \).

**Proof:** Slack variables are assumed zero and \( \lambda_j = 0, \lambda_{p} = 1 \forall j \neq p \) then \( h' = 1 \) is a feasible solution for model (11) and since objective function is maximum so, \( h \geq 1 \)

**Theorem 3.7:** In each optimal solution for model (11), \( \lambda^* \neq 0 \).

**Proof:** Same as the proof for theorem (3.2).

**Theorem 3.8:** If \( h^* \) is the optimal solution of SORM model in output oriented with integer and negative data, then \( \left(x_{ip}^{1}, x_{ip}^{2}, h^* y_{rp}, h^* y_{kp}, \frac{1}{h} y_{kp}^{2}\right) \in \partial T \).

**Proof:** Same as the proof for theorem (3.4).

### 4 Numerical example

Suppose that there are 15 DMUs with two inputs and two outputs shown in Table (1), and firstly, all inputs and outputs are true, only having integer term. Secondly, second input and second output is a positive value which does not have any sign. It means that there is a positive value for some of DMUs and a negative value for some others. Considering what was mentioned above, variables with no sign are converted to two variables shown in Table (2).

In Table (3), we have the efficiency value for these DMUs.
Conclusion and recommendation

The new model, presented in this paper is called SORM model in input oriented and output oriented with integer and negative data. This model could allocate either negative or positive data to itself and this is considered as an advantage of the provided model. But its defect is on the integer data. When numbers of integer inputs and outputs for DMUs are high, we could not solve all of the problems and furthermore, it makes the calculation process complex and difficult. It's recommended to make a new model that can, not only allocate negative and integer data to itself but also it would be able to avoid making any problems for the DMUs and also for the numbers of integer inputs and outputs, by having simple calculation process. Ultimately, it's recommended to develop a new model that could deal with negative data, under constant returns to scale.
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