Some integrals involving generalized \(k\)-Struve functions

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Abstract
The close form of some integrals involving recently developed generalized \(k\)-Struve functions is obtained. The outcome of these integrations is expressed in terms of generalized Wright functions. Several special cases are deduced which lead to some known results.

Keywords: Gamma function, \(k\)-gamma function, Lavoie-Trottier integral formula, Wright function, \(k\)-Struve function.

1 Introduction
The Struve function of order \(p\) given by
\[
H_p(x) := \sum_{k=0}^{\infty} (-1)^k \frac{(-1)^k}{\Gamma(k+3/2)\Gamma(k+p+\frac{3}{2})} \left(\frac{x}{2}\right)^{2k+p+1},
\]
(1.1)
is a particular solution of the non-homogeneous Bessel differential equation [2]
\[
x^2y''(x) + xy'(x) + (x^2 - p^2)y(x) = \frac{4(\frac{x}{2})^p}{\sqrt{\pi\Gamma(p+1/2)}}.
\]
(1.2)

Recently, in [17] Nisar et al., introduced and studied various properties of \(k\)-Struve function \(S_k(x,c)\) defined by
\[
S_k(x,c) := \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma(rk+1)\Gamma(r+\frac{3}{2})} \left(\frac{x}{2}\right)^{2r+\frac{3}{2}+1}.
\]
(1.3)
The generalized Wright hypergeometric function \(p\psi_q(z)\) is given by the series
\[
p\psi_q(z) = p\psi_q \left[\begin{array}{c} \alpha_i, \alpha_j, \beta_i, \beta_j \\ \alpha_1, \beta_1 \end{array} \right] = \sum_{k=0}^{q} \frac{\prod_{\beta_j} \Gamma(\alpha_i + \alpha_k)}{\prod_{\beta_j} \Gamma(\beta_j + \beta_k)} k!,
\]
(1.4)
where \(\alpha_i, \beta_j \in \mathbb{C}\), and real \(\alpha_i, \beta_j \in \mathbb{R}\) \((i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)\). Asymptotic behavior of this function for large values of argument of \(z \in \mathbb{C}\) were studied in [8] and under the condition
\[
\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1
\]
(1.5)

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was found in the work [19, 20]. Properties of this generalized Wright function can be seen in [10, 11, 9]. In particular, it was proved in [10] that \( p \mathcal{W}_d(z), z \in \mathbb{C} \) is an entire function under the condition (1.5).

The \( k \)-gamma function defined in [5] as:

\[
\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-t^k} dt, z \in \mathbb{C}.
\] (1.6)

By inspection the following relation holds:

\[
\Gamma_k(z + k) = z\Gamma_k(z)
\] (1.7)

and

\[
\Gamma_k(z) = k^{z-1}\Gamma\left(\frac{z}{k}\right).
\] (1.8)

If \( k \to 1 \) and \( c = 1 \), then the generalized \( k \)-Struve function defined in (1.3) reduces to the well known classical Struve function \( S_k \) defined in [7]. In this paper, we define a class of integral formulas which involves the generalized \( k \)-Struve function as defined in (1.3). Also, we investigate some special cases as the corollaries. For this purpose, we recall the following result from [13]. For \( \Re(\alpha) > 0 \) and \( \Re(\beta) > 0 \), it follows that

\[
\int_0^1 x^{a-1}(1-x)^{\beta-1}\left(1-\frac{x}{3}\right)^{2a-1}\left(1-\frac{x}{4}\right)^{\beta-1} dx = \left(\frac{2}{3}\right)^{2a} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \left(\frac{2}{3}\right)^{2a} B(\alpha, \beta).
\] (1.9)

Here \( B(\alpha, \beta) \) is the well-known beta function [18]. For more details about integral representations of various special function, the reader can refer these papers (see [3, 4, 14, 15, 16]).

2 Main Results

In this section, we establish two generalized integral formulas containing (1.3), which represented in terms of generalized Wright function defined in (1.4) by inserting with the suitable argument defined in (1.9).

**Theorem 2.1.** For \( k \in \mathbb{R}, \alpha, \mu, \beta, v, c \in \mathbb{C} \) with \( v > \frac{3}{2} k, \Re(\alpha + \mu) > 0, \Re(\alpha + \frac{\nu}{k} + 1) > 0 \) and \( x > 0 \), then the following integral formula hold true:

\[
\int_0^1 x^{\alpha+\mu-1}(1-x)^{2a-1}\left(1-\frac{x}{3}\right)^{2(\alpha+\mu)-1}\left(1-\frac{x}{4}\right)^{\alpha-1} S_{v,c}^z \left( y \left(1-\frac{x}{2}\right) \left(1-x\right)^2 \right) dx
\]

\[= \frac{(\frac{1}{2})^{\frac{2a-1}{k}} \Gamma(\alpha + \mu) (\frac{2}{3})^{2(\alpha+\mu)}}{k^{2a}} \times 2 \Psi_3 \left[ \frac{1}{4} \begin{array}{c}
(\alpha + \frac{\nu}{k} + 1, 1, 1); (1, 1); (\frac{\nu}{k} + 1, 1, 1) \times (2\alpha + \frac{\nu}{k} + \mu, 2) \end{array} \right].
\] (2.10)
Proof. Let $I$ be the left hand side of (2.1) and applying (1.3) to the integrand of (2.10), we have

$$
\int_0^1 x^{\alpha+\mu-1}(1-x)^{2\alpha-1}(1-x)\frac{\pi}{3}2(\alpha+\mu-1)(1-x)\alpha^{-1}S_{k,x,c}^e \left( \frac{\gamma(1-x^2)}{2} \right) dx
$$

$$
= \int_0^1 x^{\alpha+\mu-1}(1-x)^{2\alpha-1}(1-x)\frac{\pi}{3}2(\alpha+\mu-1)(1-x)\alpha^{-1}
\times \sum_{r=0}^\infty \frac{(-c)^r \gamma^{2r+\frac{\pi}{3}+1}}{\Gamma(kr+\nu+\frac{1}{3}k)r!}\Gamma(r+\frac{3}{2})
dx.
$$

Interchanging the order of integration and summation, which is verified by the uniform convergence of the series under the given assumption of theorem 2.1, we have

$$
I = \sum_{r=0}^\infty \frac{(-c)^r \gamma^{2r+\frac{\pi}{3}+1}}{\Gamma(kr+\nu+\frac{1}{3}k)r!}\Gamma(r+\frac{3}{2})
\times \int_0^1 x^{\alpha+\mu-1}(1-x)^{2(2r+\alpha+\frac{\nu}{k}+1)-1}(1-x)^{\gamma(2(\alpha+\mu-1)-(1-x)\alpha^{-1})(1-x)\alpha^{-1}} dx,
$$

From the assumption given in theorem 2.1, since $\Re(\frac{\nu}{k}) > 0, \Re(\alpha+\frac{\nu}{k}+1+2r) > \Re(\alpha+\frac{\nu}{k}+1) > 0, \Re(\alpha+\mu) > 0, k > 0$ and using (1.9), we get

$$
I = \sum_{r=0}^\infty \frac{(-c)^r \gamma^{2r+\frac{\pi}{3}+1}}{\Gamma(kr+\nu+\frac{1}{3}k)r!}\Gamma(r+\frac{3}{2})
\times \frac{2^{(\alpha+\mu)} \Gamma(\alpha+\mu) \Gamma(2r+\alpha+\frac{\nu}{k}+1)}{\Gamma(2r+2\alpha+\mu+\frac{\nu}{k}+1)\Gamma(kr+\nu+\frac{1}{3}k)r!}\Gamma(r+\frac{3}{2})
$$

Using (1.8), we get

$$
I = \left( \frac{2}{3} \right)^{2(\alpha+\mu)} \frac{\Gamma(\alpha+\mu) \Gamma(\frac{\nu}{k}+\frac{3}{2})}{\Gamma(\gamma(2r+2\alpha+\mu+\frac{\nu}{k}+1))}
\times \frac{\Gamma(2r+\alpha+\frac{\nu}{k}+1)}{\Gamma(kr+\nu+\frac{1}{3}k)r!}\Gamma(r+\frac{3}{2})
$$

In view of definition of Fox-Wright function (1.4), we get the required result. □

Theorem 2.2. For $k \in \mathbb{R}, \alpha, \mu, \beta, \nu, c \in \mathbb{C}$ with $\nu > \frac{1}{2}k, \Re(\alpha+\mu) > 0, \Re(\alpha+\frac{\nu}{k}+1) > 0$ and $x > 0$, then the following integral formula hold true:

$$
\int_0^1 x^{\alpha+\mu-1}(1-x)^{2(\alpha+\mu)-1}(1-x)\frac{\pi}{3}2(\alpha+\mu-1)(1-x)\alpha^{-1}S_{k,x,c}^e \left( \frac{\gamma(1-x^2)}{2} \right) dx
$$

$$
= \frac{\Gamma(\alpha+\mu) \Gamma(\frac{\nu}{k}+\frac{3}{2})}{k^\frac{1}{3} \Gamma(\gamma(2\alpha+\mu+\frac{\nu}{k}+1))}
\times \frac{(\alpha+\frac{\nu}{k}+1,2),(1,1); (\alpha+\frac{\nu}{k}+1,2),(1,1); (2\alpha+\frac{\nu}{k}+\mu,2)}{2^{(\gamma(1-x^2))} \frac{\gamma(1-x^2)}{2} \frac{\gamma(1-x^2)}{2}}.
$$

(2.11)
Proof. Let $\ell$ be the left hand side of (2.2) and applying (1.3) to the integrand of (2.10), we have

$$
\int_0^1 x^{\alpha-1} (1-x)^{2(\alpha+\mu)-1} \left(1 - \frac{x}{3}\right) 2^{\alpha-1} (1 - \frac{x}{4})^{(\alpha+\mu)-1} \left(\frac{\sqrt{x} \left(1 - \frac{3}{4}x\right)}{2}\right) dx
$$

$$
= \int_0^1 x^{\alpha-1} (1-x)^{2(\alpha+\mu)-1} (1 - \frac{x}{4})^{(\alpha+\mu)-1} \times \sum_{r=0}^{\infty} \frac{(-c)^r \left(\frac{x}{2}\right)^{2r+\frac{1}{2}+1}}{\Gamma_k(rk + \nu + \frac{3}{2}k)\Gamma(r + \frac{3}{2})} dx
$$

Interchanging the order of integration and summation, which is verified by the uniform convergence of the series under the given assumption of theorem 2.2, we have

$$
\ell = \sum_{r=0}^{\infty} \frac{(-c)^r \left(\frac{x}{2}\right)^{2r+\frac{1}{2}+1}}{\Gamma_k(rk + \nu + \frac{3}{2}k)\Gamma(r + \frac{3}{2})} \times \int_0^1 x^{(\alpha + \nu + 1 + 2r) - 1} (1-x)^{2(\alpha+\mu)-1} (1 - \frac{x}{4})^{2(\alpha + k + 1 + 2r) - 1} (1 - \frac{x}{4})^{(\alpha+\mu)-1} dx,
$$

Since $\Re\left(\frac{\nu}{2}\right) > 0$, $\Re(\alpha + \nu + 1 + 2r) > \Re(\alpha + \frac{\nu}{2} + 1) > 0$, $\Re(\alpha + \mu) > 0$, $k > 0$ and using (1.9), we get

$$
\ell = \sum_{r=0}^{\infty} \frac{(-c)^r \left(\frac{x}{2}\right)^{2r+\frac{1}{2}+1}}{\Gamma_k(rk + \nu + \frac{3}{2}k)\Gamma(r + \frac{3}{2})} \frac{\Gamma(\alpha + \mu + 1) \Gamma(2r + \alpha + \nu + 1)}{\Gamma(2r + 2\alpha + \mu + \frac{\nu}{2} + 1)} \frac{\Gamma(\alpha + \mu + 1) \Gamma(2r + \alpha + \nu + 1)}{\Gamma(2r + 2\alpha + \mu + \frac{\nu}{2} + 1)}
$$

Using (1.8), we obtain

$$
\ell = \left(\frac{2}{3}\right)^{2(\alpha + \frac{\nu}{2} + 1)} \frac{\Gamma(\alpha + \mu) \left(\frac{\nu}{2}\right)^{\frac{\nu}{2}+1}}{\Gamma(2r + 2\alpha + \mu + \frac{\nu}{2} + 1)} \frac{\Gamma(2r + \alpha + \nu + 1)}{\Gamma_k(rk + \nu + \frac{3}{2}k)\Gamma(r + \frac{3}{2})}
$$

In view of definition of Fox-Wright function (1.4), we get the desired result.

\[\square\]

3 Special Cases

In this section, we obtain the integral representation of Struve function and modified Struve function.

Case 1. If we set $c = k = 1$ in (1.3), then we get Struve function of order $\nu$ as

$$
H_{\nu}(x) := \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r+\nu+1}}{\Gamma(r + \nu + \frac{3}{2})\Gamma(r + \frac{3}{2})}.
$$

Case 2. If we set $c = -1$ and $k = 1$ in (1.3), then we get modified Struve function,

$$
L_{\nu}(x) := \sum_{r=0}^{\infty} \frac{1}{\Gamma(r + \nu + \frac{3}{2})\Gamma(r + \frac{3}{2})} \left(\frac{x}{2}\right)^{2r-\nu+1}.
$$

Corollary 3.1. Suppose that the conditions of Theorem 2.1 are satisfied. Then the following integral formula holds:

$$
\int_0^1 x^{\alpha+\mu-1} (1-x)^{2(\alpha+\mu)-1} (1 - \frac{x}{3})^{2(\alpha+\mu)-1} (1 - \frac{x}{4})^{(\alpha+\mu)-1} H_{\nu} \left(\frac{\sqrt{x} \left(1 - \frac{3}{4}x\right)}{2}\right) dx
$$
\[
\begin{align*}
&= \left(\frac{\gamma}{2}\right)^{\nu+1} \Gamma\left(\alpha + \mu\right) \left(\frac{2}{3}\right)^{2(\nu+\alpha)} \\
&\times 2^\Psi_3 \left[ \begin{array}{c}
(\alpha + \nu + 1, 2), (1, 1); \\
(v + \frac{3}{2}, 1), (\frac{3}{2}, 1), (2\alpha + \nu + \mu, 2)
\end{array} \right]. (3.14)
\end{align*}
\]

**Corollary 3.2.** Assume that the conditions of theorem 2.2 satisfied, then the following integral formula hold true:

\[
\int_0^1 x^{\alpha-1} \left(1-x\right)^{2(\alpha+\mu)-1} \left(1-\frac{x}{3}\right)^{2\alpha-1} \left(1-\frac{x}{4}\right) (\alpha+\mu+1) L_\nu \left(\frac{\sqrt{-1}}{2}\right) dx
\]

\[
= \left(\frac{\gamma}{2}\right)^{\nu+1} \Gamma\left(\alpha + \mu\right) \left(\frac{2}{3}\right)^{2\alpha} \\
\times 2^\Psi_3 \left[ \begin{array}{c}
(\alpha + \nu + 1, 2), (1, 1); \\
(v + \frac{3}{2}, 1), (\frac{3}{2}, 1), (2\alpha + \nu + \mu, 2)
\end{array} \right]. (3.15)
\]

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