Exact solutions for the family of third order Korteweg de-Vries equations

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Abstract
In this work we apply an extended hyperbolic function method to solve the nonlinear family of third order Korteweg de-Vries (KdV) equations, namely, the KdV equation, the modified KdV (mKdV) equation, the potential KdV (pKdV) equation, the generalized KdV (gKdV) equation and gKdV with two power nonlinearities equation. New exact travelling wave solutions are obtained for the KdV, mKdV and pKdV equations. The solutions are expressed in terms of hyperbolic functions, trigonometric functions and rational functions. The method used is promising method to solve other nonlinear evaluation equations.

Keywords: Nonlinear family of third order Korteweg de-Vries, KdV equation, mKdV equation, pKdV equation, gKdV equation, gKdV with two power nonlinearities, Extended hyperbolic function method, Ordinary differential equations, Travelling wave solutions.

1 Introduction

Nonlinear partial differential equations (PDE’s) have a significant role in several scientific and engineering fields. Since the discovery of soliton in 1965 by Zabusky and Kruskal [1], many nonlinear PDE’s have been derived and extensively applied in different branches of physics and applied mathematics. Nonlinear PDE’s appear in condensed matter, solid state physics, fluid mechanics, chemical kinetics, plasma physics, nonlinear optics, propagation of fluxions in Josephson junctions, theory of turbulence, ocean dynamics, biophysics and star formation and many others. In order to understand the different nonlinear phenomena, various methods for obtaining exact solutions to nonlinear PDE’s have been proposed. Among these are the inverse scattering method [2], Hirota’s method [3], Backlund transformation [4], F-expansion method [5], homogeneous balance method [6], tanh-function method [7], Jacobi elliptic function method [8], and many others.

In this paper, we consider the nonlinear family of third order Korteweg de-Vries (KdV) equations in [9] of the form

$$u_t + P(u)u_x + u_{xxx} = 0$$

(1.1)

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where \( u(x,t) \) is a function of space \( x \) and time \( t \). The nonlinear term \( P(u) \) takes the forms

\[
P(u) = \begin{cases} 
    au \\
    au^2 \\
    u_x \\
    au^2 \\
    au^n - bu^2n.
\end{cases}
\]  

For \( P(u) = au \), Eq. (1.1) becomes the standard KdV equation. When \( P(u) = au^2 \), Eq. (1.1) is called the mKdV equation [10]. For \( P(u) = au^n \), Eq. (1.1) is called the gKdV equation [11]. If \( P(u) = u_x \), then Eq. (1.1) is called pKdV equation. When \( P(u) = au^n - bu^2n \), we get the generalized KdV equation with two power nonlinearities [12]. The KdV equation is used to model the disturbance of the surface of shallow water in the presence of solitary waves. It is a generic model for the study of weakly nonlinear long waves, incorporating leading order nonlinearity and dispersion [13]. The mKdV equation appears in electric circuits and multi-component plasmas [14]. The generalized KdV equation serves as an approximate model for the description of weak dispersive effects on the propagation of nonlinear waves characteristic direction [14]. The nonlinear family of third order Korteweg de-Vries equations and other related equations were solved by many methods. For single soliton solutions, the tanh-function method [15], the tanh-coth method [16], the sine-cosine method [17], the inverse scattering method [18], were used. For the concept of multiple solitions solutions, the Hirota bilinear formalism [19] and a simplified version of it [20] were used. In this work, an extended hyperbolic function method [21-25] is used to handle the nonlinear family of third order Korteweg de-Vries equations, namely, the KdV, mKdV, pKdV, gKdV and gKdV with two power nonlinearities equations. As a result, new exact travelling wave solutions are obtained for the KdV, mKdV and pKdV equations. The solutions are expressed in terms of hyperbolic functions, trigonometric functions and rational functions.

2 The Extended Hyperbolic Function Method

In this section, we use a general method, namely, the extended hyperbolic function method [25], to obtain multiple exact special solutions for nonlinear wave equations. The solitary wave solutions, the singular traveling wave solutions, periodic wave solutions of triangle function types, and traveling wave solutions of rational function type are constructed uniformly by this method. Given nonlinear partial differential equation, for instance, in two variables, as follows

\[
P(u, u_x, u_{xx}, u_{xxt}, \ldots) = 0
\]  

where \( P \) is in general a nonlinear function of its arguments, the subscripts denote the partial derivatives. Let \( u(x,t) = u(\xi), \xi = x - ct \), then Eq. (2.3) reduces to a nonlinear ordinary differential equation (ODE)

\[
Q(u, u', u'', \ldots) = 0.
\]  

We suppose that the solution of the ODE (2.4) is of the form

\[
u(x,t) = u(\xi) = a_0 + \sum_{i=1}^{n} a_i (v(\xi))^i + \sum_{j=1}^{n} b_j (v(\xi))^{-j},
\]  

where the coefficients \( a_0, a_i, b_j (i = 1, 2, \ldots n ; j = 1, 2, \ldots n) \) are constants to be determined and \( v = v(\xi) \) satisfies a nonlinear ordinary differential equation of first order

\[
v' = \frac{dv}{d\xi} = d + ev^2(\xi),
\]  

The degree \( n \) of the polynomial in (2.5) can be determined via balancing the highest order derivative terms and the nonlinear term in ODE. Substituting (2.5) into (2.4), using (2.6) repeatedly, and setting the coefficients of the each order of \( v' \) to zero, we obtain a set of nonlinear algebraic equations for \( a_0, a_i, b_j (i = 1, 2, \ldots n ; j = 1, 2, \ldots n), d, e, c. \)
With the aid of the computer program Maple, we can solve the set of nonlinear algebraic equations and obtain all the constants \(a_0, a_i, b_j (i = 1, 2, \ldots; j = 1, 2, \ldots, n), d, e, c\). The ODE (2.6) has the following six kinds of general solutions

\[
\begin{align*}
    v(\xi) &= \text{sgn}(d) \sqrt{\frac{d}{e}} \tan \left( \sqrt{\frac{d}{e}} \xi \right), \quad de > 0 \\
    v(\xi) &= -\text{sgn}(d) \sqrt{\frac{d}{e}} \cot \left( \sqrt{\frac{d}{e}} \xi \right), \quad de > 0 \\
    v(\xi) &= \text{sgn}(d) \sqrt{-\frac{d}{e}} \tanh \left( \sqrt{-\frac{d}{e}} \xi \right), \quad de < 0 \\
    v(\xi) &= \text{sgn}(d) \sqrt{-\frac{d}{e}} \coth \left( \sqrt{-\frac{d}{e}} \xi \right), \quad de < 0 \\
    v(\xi) &= -\frac{1}{e^2}, \quad d = 0, e > 0 \\
    v(\xi) &= d \xi, \quad d \in \mathbb{R}, e = 0.
\end{align*}
\]  

(2.7)

The multiple exact special solutions of nonlinear partial differential equation (2.3) are obtained by making use of (2.5) and (2.7).

### 2.1 Applications of the Extended Hyperbolic Function Method

#### 2.1.1 The KdV equation

\[
u_t + auu_x + u_{xxx} = 0. \tag{2.8}
\]

Here we choose \(P(u) = au\) in (1.1). Substituting \(u(x,t) = u(\xi), \xi = x - ct\), into Eq. (2.8) and integrating once yields

\[
-cu + \frac{a}{2}u^2 + u'' = 0. \tag{2.9}
\]

Balancing the order of the nonlinear term \(u^2\) with the highest derivative \(u''\) gives \(2n = n + 2\) that gives \(n = 2\). Thus, the solution of (2.9) has the form

\[
u(\xi) = a_0 + a_1v(\xi) + a_2v^2(\xi) + b_1v^{-1}(\xi) + b_2v^{-2}(\xi). \tag{2.10}
\]

Substituting (2.10) in (2.9) and using (2.6), collecting the coefficients of each power of \(v^i, 0 \leq i \leq 8\), setting each coefficient to zero, and solving the resulting system obtain the following sets of solutions

- \(a_0 = -\frac{4de}{a}, a_1 = b_1 = 0, a_2 = -\frac{12e^2}{a}, b_2 = 0, c = 4de\)
- \(a_0 = -\frac{12de}{a}, a_1 = b_1 = 0, a_2 = -\frac{12e^2}{a}, b_2 = 0, c = -4de\)
- \(a_0 = -\frac{4de}{a}, a_1 = b_1 = 0, a_2 = 0, b_2 = -\frac{12e^2}{a}, c = 4de\)
- \(a_0 = -\frac{12de}{a}, a_1 = b_1 = 0, a_2 = 0, b_2 = -\frac{12e^2}{a}, c = -4de\)
- \(a_0 = -\frac{8de}{a}, a_1 = b_1 = 0, a_2 = -\frac{8e^2}{a}, b_2 = -\frac{12e^2}{a}, c = 16de\)
- \(a_0 = -\frac{24de}{a}, a_1 = b_1 = 0, a_2 = -\frac{12e^2}{a}, b_2 = -\frac{12e^2}{a}, c = -16de\).

For \(c > 0\), using (2.10), (2.7) and the above sets of solutions, we get

\[
u_t(x,t) = \frac{c}{a} - \frac{3c}{a} \tan^2 \left( \frac{\sqrt{c}}{2} (x - ct) \right) = \frac{c}{a} \left[ 1 + 3 \tan^2 \left( \frac{\sqrt{c}}{2} (x - ct) \right) \right],
\]
\[ u_2(x,t) = \frac{3c}{a} - \frac{3c}{a} \tanh^2 \left( \frac{\sqrt{c}}{2} (x - ct) \right) \]
\[ = \frac{3c}{a} \sech^2 \left( \frac{\sqrt{c}}{2} (x - ct) \right) , \]
\[ u_4(x,t) = -\frac{c}{a} \left[ 1 + \cot^2 \left( \frac{\sqrt{c}}{2} (x - ct) \right) \right] , \]
\[ u_4(x,t) = \frac{3c}{a} - \frac{3c}{a} \coth^2 \left( \frac{\sqrt{c}}{2} (x - ct) \right) \]
\[ = -\frac{3c}{a} \csch^2 \left( \frac{\sqrt{c}}{2} (x - ct) \right) , \]
\[ u_5(x,t) = \frac{3c}{4a} \left[ 2 - \tanh^2 \left( \frac{\sqrt{c}}{4} (x - ct) \right) - \coth^2 \left( \frac{\sqrt{c}}{4} (x - ct) \right) \right] , \]
\[ u_6(x,t) = \frac{c}{4a} \left[ 2 - 3\tanh^2 \left( \frac{\sqrt{c}}{4} (x - ct) \right) - 3\cot^2 \left( \frac{\sqrt{c}}{4} (x - ct) \right) \right] . \]

If we choose \( a = 1 \) in (2.8) then
\[ u_7(x,t) = \frac{-12}{x^2} . \]
Also if we choose \( a = -1 \) in (2.8) then
\[ u_8(x,t) = \frac{12}{x^2} . \]

For \( c < 0 \)
\[ u_9(x,t) = -\frac{c}{a} \left[ 1 - 3\tanh^2 \left( \frac{\sqrt{-c}}{2} (x - ct) \right) \right] , \]
\[ u_{10}(x,t) = \frac{3c}{a} \sec^2 \left( \frac{\sqrt{-c}}{2} (x - ct) \right) , \]
\[ u_{11}(x,t) = -\frac{c}{a} \left[ 1 - 3\coth^2 \left( \frac{\sqrt{-c}}{2} (x - ct) \right) \right] , \]
\[ u_{12}(x,t) = \frac{3c}{a} \csch^2 \left( \frac{\sqrt{-c}}{2} (x - ct) \right) , \]
\[ u_{13}(x,t) = \frac{3c}{4a} \left[ 2 + \tanh^2 \left( \frac{\sqrt{-c}}{4} (x - ct) \right) + \cot^2 \left( \frac{\sqrt{-c}}{4} (x - ct) \right) \right] , \]
\[ u_{14}(x,t) = \frac{c}{4a} \left[ 2 + 3\tanh^2 \left( \frac{\sqrt{-c}}{4} (x - ct) \right) + 3\cot^2 \left( \frac{\sqrt{-c}}{4} (x - ct) \right) \right] . \]

\( u_9, u_7 \) and \( u_8 \) are new solutions. Other solutions were obtained by Wazwaz using the tanh-coth method in [14], see pages 497–499.

2.1.2 The mKdV equation

\[ u_t + au^2 u_x + u_{xxx} = 0 . \] (2.11)

Here we choose \( P(u) = au^2 \) in (1.1). Substituting \( u(x,t) = u(\xi) , \xi = x - ct \), into Eq. (2.11) and integrating once yields
\[ -cu + \frac{a}{3} u^3 + u'' = 0 . \] (2.12)
Balancing the order of the nonlinear term \(u^3\) with the highest derivative \(u''\) gives \(3n = n + 2\) that gives \(n = 1\). Thus, the solution of (2.12) has the form

\[
u(\xi) = a_0 + a_1 v(\xi) + b_1 v^{-1}(\xi).
\]  
(2.13)

Substituting (2.13) in (2.12) and using (2.6), collecting the coefficients of each power of \(\nu\), \(0 \leq i \leq 6\), setting each coefficient to zero, and solving the resulting system obtain the following sets of solutions

- \(a_0 = a_1 = 0, b_1 = \pm \sqrt{-\frac{6}{\pi}} d, c = 2de, e = e, d = d\)
- \(a_0 = b_1 = 0, a_1 = \pm \sqrt{-\frac{6}{\pi}} e, c = 2de, e = e, d = d\)
- \(a_0 = 0, a_1 = \pm \sqrt{-\frac{6}{\pi}} e, b_1 = \pm \sqrt{-\frac{6}{\pi}} d, c = -4de, e = e, d = d\)
- \(a_0 = 0, a_1 = \pm \sqrt{-\frac{6}{\pi}} e, b_1 = \mp \sqrt{-\frac{6}{\pi}} d, c = 8de, e = e, d = d\)

For \(c > 0, a < 0\), using (2.13)(2.7) and the above sets of solutions we get

\[
u_{12}(x,t) = \pm \sqrt{-\frac{3c}{a}} \tan \left( \sqrt{\frac{c}{2}} (x - ct) \right),
\]
\[
u_{13,4}(x,t) = \pm \sqrt{-\frac{3c}{a}} \cot \left( \sqrt{\frac{c}{2}} (x - ct) \right),
\]
\[
u_{15,8}(x,t) = \sqrt{\frac{3c}{2a}} \left[ \pm \tanh \left( \frac{\sqrt{c}}{2} (x - ct) \right) \pm \coth \left( \frac{\sqrt{c}}{2} (x - ct) \right) \right],
\]
\[
u_{17,8}(x,t) = \frac{1}{2} \sqrt{-\frac{3c}{a}} \left[ \pm \tan \left( \frac{1}{2} \sqrt{\frac{c}{2}} (x - ct) \right) \pm \cot \left( \frac{1}{2} \sqrt{\frac{c}{2}} (x - ct) \right) \right].
\]

For \(c < 0, a < 0\)

\[
u_{9,10}(x,t) = \pm \sqrt{\frac{3c}{a}} \tanh \left( \sqrt{\frac{-c}{2}} (x - ct) \right),
\]
\[
u_{11,14}(x,t) = \pm \sqrt{\frac{3c}{a}} \coth \left( \sqrt{\frac{-c}{2}} (x - ct) \right),
\]
\[
u_{13,14}(x,t) = \sqrt{\frac{3c}{2a}} \left[ \pm \tanh \left( \frac{\sqrt{-c}}{2} (x - ct) \right) \mp \coth \left( \frac{\sqrt{-c}}{2} (x - ct) \right) \right],
\]
\[
u_{15,16}(x,t) = \frac{1}{2} \sqrt{\frac{3c}{a}} \left[ \pm \tanh \left( \frac{1}{2} \sqrt{\frac{-c}{2}} (x - ct) \right) \mp \coth \left( \frac{1}{2} \sqrt{\frac{-c}{2}} (x - ct) \right) \right].
\]

If we choose \(a = -6\) in Eq. (2.11) then

\[
u_{17,18}(x,t) = \pm \frac{1}{x}.
\]

Also if we choose \(a = 6\) in Eq. (2.11) then

\[
u_{19,20}(x,t) = \pm \frac{i}{x}.
\]

If we replace Eq. (2.6) by

\[v' = \frac{dv}{d\xi} = v(\xi) \sqrt{d + e v^2(\xi)}.\]
Then we get
\[
\begin{align*}
  u_{21.22}(x,t) &= \pm \sqrt{-\frac{6c}{a}} \operatorname{csch} \left( \sqrt{c} (x - ct) \right), \quad c > 0, a < 0 \\
  u_{23.24}(x,t) &= \pm \sqrt{\frac{6c}{a}} \operatorname{sech} \left( \sqrt{c} (x - ct) \right), \quad c > 0, a > 0 \\
  u_{25.26}(x,t) &= \pm \sqrt{\frac{6c}{a}} \operatorname{csc} \left( \sqrt{-c} (x - ct) \right), \quad c < 0, a < 0 \\
  u_{27.28}(x,t) &= \pm \sqrt{\frac{6c}{a}} \operatorname{sec} \left( \sqrt{-c} (x - ct) \right), \quad c < 0, a > 0
\end{align*}
\]

\(u_{5,6,7,8, u_{17,18}}\) and \(u_{19,20}\) are new solutions. Other solutions were obtained by Wazwaz using the tanh-coth method in [14], see pages 508-509.

2.1.3 The pKdV equation
\[
u_t + a(u_x)^2 + u_{xxx} = 0. \tag{2.14}
\]
Here we choose \(P(u) = au_x\) in (1.1). Substituting \(u(x,t) = u(\xi), \xi = x - ct\), into Eq. (2.14) we get
\[
-cu_t + a(u_x)^2 + u''' = 0. \tag{2.15}
\]
Balancing the order of the nonlinear term \((u_x)^2\) with the highest derivative \(u'''\) gives \(2(n + 1) = n + 3\) that gives \(n = 1\). Thus, the solution of (2.15) has the form
\[
u(\xi) = a_0 + a_1v(\xi) + b_1v^{-1}(\xi). \tag{2.16}
\]
Substituting (2.16) in (2.15) and using (2.6), collecting the coefficients of each power of \(v^i, \ i = 2m, 0 \leq m \leq 4\), setting each coefficient to zero, and solving the resulting system obtain the following sets of solutions

- \(a_0 = a_0, a_1 = a_1, b_1 = 0, c = \frac{2}{a} da_1, e = -\frac{1}{6} aa_1, d = d\)
- \(a_0 = a_0, a_1 = 0, b_1 = b_1, c = -ab_1 e, e = e, d = 0\)
- \(a_0 = a_0, a_1 = 0, b_1 = b_1, c = -\frac{3}{2} ab_1 e, e = e, d = \frac{1}{6} ab_1\)
- \(a_0 = a_0, a_1 = a_1, b_1 = b_1, c = \frac{4}{a^2} a_1 b_1 e, e = -\frac{1}{6} aa_1, d = \frac{1}{6} ab_1\)

For \(c > 0\), using (2.16), (2.7) and the above sets of solutions we get
\[
\begin{align*}
  u_1(x,t) &= a_0 + \frac{c}{a} (x - ct), \quad \forall a_0 \in \mathbb{R}, \\
  u_2(x,t) &= a_0 + 3\frac{\sqrt{c}}{a} \tanh \left( \frac{\sqrt{c}}{2} (x - ct) \right), \\
  u_3(x,t) &= a_0 + 3\frac{\sqrt{c}}{a} \coth \left( \frac{\sqrt{c}}{2} (x - ct) \right), \\
  u_4(x,t) &= a_0 + \frac{3\sqrt{c}}{2a} \left[ \tanh \left( \frac{\sqrt{c}}{4} (x - ct) \right) + \coth \left( \frac{\sqrt{c}}{4} (x - ct) \right) \right].
\end{align*}
\]

For \(c < 0\)
\[
\begin{align*}
  u_5(x,t) &= a_0 - 3\frac{\sqrt{-c}}{a} \tan \left( \frac{\sqrt{-c}}{2} (x - ct) \right), \forall a_0 \in \mathbb{R}, \\
  u_6(x,t) &= a_0 + 3\frac{\sqrt{-c}}{a} \cot \left( \frac{\sqrt{-c}}{2} (x - ct) \right).
\end{align*}
\]
Here we choose $P(u) = au^n$ in (1.1). Substituting $u(x,t) = u(\xi)$, $\xi = x - ct$, into Eq. (2.17) and integrating once yields
\begin{equation}
-cu + \frac{a}{n+1} u^{n+1} + u'' = 0.
\end{equation}
Balancing the order of the nonlinear term $u^{n+1}$ with the highest derivative $u''$ gives $(n+1)m = n+2$ that gives $m = \frac{2}{5}$, $m$ should be integer, then we use the transformation
\begin{equation}
u(\xi) = u^{\frac{1}{5}}(\xi).
\end{equation}
Substituting (2.19) in (2.18) we get
\begin{equation}
-c n + 2(n+1) w^2 + an^2 w^3 + n(n+1) w w'' + \left(1-n^2\right) (w')^2 = 0.
\end{equation}
Balancing the order of $w^3$ with $w w''$ gives $3m = m + m + 2$ that gives $m = 2$. Thus, the solution of (2.20) has the form
\begin{equation}
 w(\xi) = a_0 + a_1 v(\xi) + a_2 v^2(\xi) + b_1 v^{-1}(\xi) + b_2 v^{-2}(\xi).
\end{equation}
Substituting (2.21) in (2.20) and using (2.6), collecting the coefficients of each power of $v$, $0 \leq i \leq 12$, setting each coefficient to zero, and solving the resulting system obtain the following sets of solutions
\begin{itemize}
  \item $a_0 = -\frac{2de(n+1)(n+2)}{an^2}$, $a_1 = b_1 = 0$, $a_2 = -\frac{2d^2(n+1)(n+2)}{an}$, $b_2 = 0$,
  \[c = -\frac{4de}{an}, d = d, e = e\]
  \item $a_0 = -\frac{2de(n+1)(n+2)}{an^2}$, $a_1 = b_1 = 0$, $a_2 = 0$, $b_2 = -\frac{2d^2(n+1)(n+2)}{an}$,
  \[c = -\frac{4de}{an}, d = d, e = e\]
  \item $a_0 = -\frac{2de(n+1)(n+2)}{an^2}$, $a_1 = b_1 = 0$, $a_2 = 2\frac{d^2(n+1)(n+2)}{an}$, \[b_2 = \frac{2d^2(n+1)(n+2)}{an^2},
  \[c = -\frac{16de}{an}, d = d, e = e\]
\end{itemize}
For $c > 0$, using (2.21),(2.7) and the above sets of solutions we get
\begin{equation}
u_1(x,t) = \left[\frac{C}{2a} (n+1)(n+2) \text{sech}^2 \left(\frac{n}{2}\sqrt{c} (x-ct)\right)\right]^\frac{1}{2},
\end{equation}
\begin{equation}
u_2(x,t) = \left[\frac{-C}{2a} (n+1)(n+2) \text{csch}^2 \left(\frac{n}{2}\sqrt{c} (x-ct)\right)\right]^\frac{1}{2},
\end{equation}
\begin{equation}
u_3(x,t) = \left[\frac{C}{8a} (n+1)(n+2) \left(2 - \text{tanh}^2 \left(\frac{n}{4}\sqrt{c} (x-ct)\right)\right) - \coth^2 \left(\frac{n}{4}\sqrt{c} (x-ct)\right)\right]^\frac{1}{2}.
\end{equation}
For $c < 0$
\begin{equation}
u_4(x,t) = \left[\frac{k}{2a} (n+1)(n+2) \text{sec}^2 \left(\frac{n}{2}\sqrt{-c} (x-ct)\right)\right]^\frac{1}{2},
\end{equation}
\begin{equation}
u_5(x,t) = \left[\frac{k}{2a} (n+1)(n+2) \text{csc}^2 \left(\frac{n}{2}\sqrt{-c} (x-ct)\right)\right]^\frac{1}{2},
\end{equation}
\begin{equation}
u_6(x,t) = \left[\frac{k}{16a} (n+1)(n+2) \left(2 + \text{tan}^2 \left(\frac{n}{4}\sqrt{-c} (x-ct)\right)\right) + \cot^2 \left(\frac{n}{4}\sqrt{-c} (x-ct)\right)\right]^\frac{1}{2}.
\end{equation}
These solutions were obtained by Wazwaz using the tanh-coth method in [14], see pages 523-525.
2.1.5 The gKdV equation with two power nonlinearities

\[ u_t + (au^p - bu^{2n}) u_x + u_{xxx} = 0. \]  

(2.22)

Here we choose \( P(u) = au^p - bu^{2n} \) in (1.1). Substituting \( u(x, t) = u(\xi), \xi = x - ct \), into Eq. (2.22) we get

\[ -cu + \frac{a}{n+1} u^{n+1} - \frac{b}{2n+1} u^{2n+1} + u'' = 0. \]  

(2.23)

Balancing the order of the nonlinear term \( u^{2n+1} \) we get \( u'' \) gives \((2n+1)m = n + 2\) that gives \( m = \frac{1}{b}, m \) should be integer, then we use the transformation

\[ u(\xi) = w^\frac{1}{2}(\xi). \]  

(2.24)

Substituting (2.24) in (2.23) we get

\[ -c n^2 (n+1)(2n+1) w^2 + an^2 (2n+1) w^3 - bn^2 (n+1) w^4 \]

\[ + n(2n+1) w^{2n} + (2n+1) (1-n^2)(w')^2 = 0. \]  

(2.25)

Balancing the order of \( w^4 \) with \( w^{2n} \) gives \( 4m = m + m + 2 \) that gives \( m = 1 \). Thus, the solution of (2.25) has the form

\[ w(\xi) = a_0 + a_1 w(\xi). \]  

(2.26)

Substituting (2.26) in (2.25) and using (2.6), collecting the coefficients of each power of \( v^i, 0 \leq i \leq 8 \), setting each coefficient to zero, and solving the resulting system obtain the set of solution

\[ a_0 = \frac{a(2n+1)^2}{2b(n+2)}, a_1 = \pm \sqrt{\frac{2n+1}{4(n+1)(n+2)}}, d = -\frac{a^2 n^2(2n+1)^3}{b(n+1)(n+2)^2}, c = \frac{a^2(2n+1)^2}{b(n+1)(n+2)} \]

For \( b > 0 \), using (2.26), (2.7) and the above set of solution we get

\[ u_{1,2}(x, t) = \left[ \frac{a(2n+1)}{2b(n+2)} \left( 1 \pm \tanh \left( \frac{a}{2(n+2)} \sqrt{\frac{2n+1}{b(n+1)}} \left( x - \frac{a^2(2n+1)}{b(n+1)(n+2)^2} t \right) \right) \right) \right]^\frac{1}{2}, \]

\[ u(x, t) = w(x, t)^\frac{1}{2}, \]

\[ u_{3,4}(x, t) = \left[ \frac{a(2n+1)}{2b(n+2)} \left( 1 \pm \coth \left( \frac{a}{2(n+2)} \sqrt{\frac{2n+1}{b(n+1)}} \left( x - \frac{a^2(2n+1)}{b(n+1)(n+2)^2} t \right) \right) \right) \right]^\frac{1}{2}. \]

For \( b < 0 \)

\[ u_{5,6}(x, t) = \left[ \frac{a(2n+1)}{2b(n+2)} \left( 1 \pm i \tan \left( \frac{a}{2(n+2)} \sqrt{-\frac{2n+1}{-b(n+1)}} \left( x - \frac{a^2(2n+1)}{b(n+1)(n+2)^2} t \right) \right) \right) \right]^\frac{1}{2}, \]

\[ u_{7,8}(x, t) = \left[ \frac{a(2n+1)}{2b(n+2)} \left( 1 \pm i \cot \left( \frac{a}{2(n+2)} \sqrt{-\frac{2n+1}{-b(n+1)}} \left( x - \frac{a^2(2n+1)}{b(n+1)(n+2)^2} t \right) \right) \right) \right]^\frac{1}{2}. \]

These solutions were obtained by Wazwaz in using the tanh method in [14], see pages 531-533.
3 Conclusion

In this article, the extended hyperbolic function method has been successfully implemented to find new traveling wave solutions for some of the nonlinear family of third order Korteweg de-Vries. The results show that this method is a powerful mathematical tool for obtaining exact solutions for the nonlinear PDE’s. It is also a promising method to solve other nonlinear PDE’s.

Acknowledgements

This project was founded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant no. (429/091-3). The authors, therefore, acknowledge with thanks DSR technical and financial support. Also, the author thanks referee and editor for their useful technical comments and valuable suggestions to improve the readability of the paper, which led to a significant improvement of the paper.

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