Analytic approximate solution for some integral equations by optimal homotopy analysis transform method

Mohamed S. Mohamed\textsuperscript{1,2*}, Muteb R. Alharthī\textsuperscript{1}, Refah A. Alotabi\textsuperscript{1}

\textsuperscript{(1)} Mathematics Department, Faculty of Science, Taif University, Taif, Saudi Arabia
\textsuperscript{(2)} Mathematics Department, Faculty of Science, Al-Azhar University, Cairo, Egypt

Abstract

The main aim of this paper is to propose a new and simple algorithm namely homotopy analysis transform method (HATM), to obtain approximate analytical solutions of integral equations. Integral equation occurs in the mathematical modeling of several models in physics, astrophysics, solid mechanics and applied sciences. The numerical solutions obtained by proposed method indicate that the approach is easy to implement and computationally very attractive. Finally, several numerical examples are given to illustrate the accuracy and stability of this method. Comparison of the approximate solution with the exact solutions also we show that the proposed method is very efficient and computationally attractive.

A new efficient approach is proposed to obtain the optimal value of convergence controller parameter $\hbar$ to guarantee the convergence of the obtained series solution.

Keywords: integral equation; optimal homotopy analysis transform method; Laplace transform.

1 Introduction

An integral equation is defined as an equation in which the unknown function $y(x)$ to be determined appear under the integral sign. The subject of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations. Abel's equation is one of the integral equations derived directly from a concrete problem of physics, without passing through a differential equation. This integral equation occurs in the mathematical modeling of several models in physics, astrophysics, solid mechanics and applied sciences. The great mathematician Niels Abel, gave the initiative of integral equations in 1823 in his study of mathematical physics [1-4]. In 1924, generalized Abel's integral equation on a finite segment was studied by Zeilon [5]. The different types of Abel integral
equation in physics have been solved by Pandey et al. [6], Kumar and Singh [7], Kumar et al. [8], Dixit et al. [9], Yousefi [10], Khan and Gondal [11], Li and Zhao [12] by applying various kinds of analytical and numerical methods. The development of science has led to the formation of many physical laws, which, when restated in mathematical form, often appear as differential equations. Engineering problems can be mathematically described by differential equations, and thus differential equations play very important roles in the solution of practical problems. For example, Newton’s law, stating that the rate of change of the momentum of a particle is equal to the force acting on it, can be translated into mathematical language as a differential equation. Similarly, problems arising in electric circuits, chemical kinetics, and transfer of heat in a medium can all be represented mathematically as differential equations.

The main aim of this article is to present analytical and approximate solution of integral equations by using new mathematical tool like optimal homotopy analysis transform method. The proposed method is coupling of the homotopy analysis method HAM and Laplace transform method. The HAM, first proposed in 1992 by Liao, has been successfully applied to solve many problems in physics and science [13-18]. In recent years many authors have paid attention to study the solutions of linear and nonlinear partial differential equations by using various methods combined with the Laplace transform [19-27].

A typical form of an integral equation in \( y(x) \) is of the form:

\[
y(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x, t)y(t)dt,
\]

(1.1)

where \( K(x, t) \) is called the kernel of the integral equation (1.1), and \( \alpha(x) \) and \( \beta(x) \) are the limits of integration. It can be easily observed that the unknown function \( y(x) \) appears under the integral sign. It is to be noted here that both the kernel \( K(x, t) \) and the function \( f(x) \) in equation (1.1) are given functions; and \( \lambda \) is a constant parameter. The prime objective of this text is to determine the unknown function \( y(x) \) that will satisfy equation (1.1) using a number of solution techniques. We shall devote considerable efforts in exploring these methods to find solutions of the unknown function.

2 Preliminaries and notations

In order to elucidate the solution procedure of the optimal homotopy analysis transform method, we consider the following integral equations of second kind:

\[
y(x) = f(x) + \int_{0}^{x} K(x, t)y(t)dt, \quad 0 \leq x \leq 1 \tag{2.2}
\]

Now operating the Laplace transform on both side in Eq. (2.2) we get

\[
L[y(x)] = L[f(x)] + L \left[ \int_{0}^{x} K(x, t)y(t)dt \right] \tag{2.3}
\]

We define the nonlinear operator

\[
N[\phi(x; q)] = L[\phi(x; q)] - L[f(x)] - L \left[ \int_{0}^{x} K(x, t)\phi(x; q)dt \right] \tag{2.4}
\]

where \( q \in [0, 1] \) be an embedding parameter and \( \phi(x; q) \) is the real function of \( x \) and \( q \). By means of generalizing the traditional homotopy methods, the great mathematician Liao [13-14] construct the zero order deformation equation

\[
(1-q)L[\phi(x; q) - y_0(x)] = hqH(x)N[\phi(x; q)], \tag{2.5}
\]

where is a nonzero auxiliary parameter, \( H(x) \neq 0 \) an auxiliary function, \( y_0(x) \) is an initial guess of \( y(x) \) and \( \phi(x; q) \) is an unknown function. It is important that one has great freedom to choose auxiliary thing in HATM. Obviously, when \( q = 0 \) and \( q = 1 \), it holds
\( \Phi(x; 0) = y_0(x), \Phi(x; 1) = y(x) \),

\( \Phi(x; q) = y_0(x, t) + \sum_{m=1}^{\infty} q^m y_m(x), \tag{2.7} \)

where

\[ y_m(x) = \frac{1}{m!} \frac{\partial^m \Phi(x; q)}{\partial q^m} \bigg|_{q=0} \tag{2.8} \]

Expanding \( \Phi(x; q) \) in Taylor's series with respect to \( q \), we have

\( \Phi(x; q) = y_0(x, t) + \sum_{m=1}^{\infty} q^m y_m(x), \)

\[ y_m(x) = \frac{1}{m!} \frac{\partial^m \Phi(x; q)}{\partial q^m} \bigg|_{q=0} \]

If the auxiliary linear operator, the initial guess, the auxiliary parameter \( \lambda \), and the auxiliary function are properly chosen, the series (2.7) converges at \( q = 1 \), we have

\( y(x) = y_0(x) + \sum_{m=1}^{\infty} y_m(x), \) \( \text{which must be one of the solutions of the original integral equations. Define the vectors} \)

\( \tilde{y}_m = \{y_0(x), y_1(x),..., y_m(x)\}. \tag{2.10} \)

Differentiating equation (2.6) \( m \) -times with respect to the embedding parameter \( q \), then setting \( q = 0 \) and finally dividing them by \( m! \), we obtain the \( m^{th} \)-order deformation equation.

\[ L[y_m(x) - \chi_m y_{m-1}(x)] = h q H(x) R_m(\tilde{y}_{m-1}, x) \tag{2.11} \]

where

\[ R_m(\tilde{y}_{m-1}, x) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \Phi(x; q)}{\partial q^{m-1}} \bigg|_{q=0} \tag{2.12} \]

and

\[ \chi_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases} \tag{2.13} \]

In this way, it is easily to obtain \( y_m(x) \) for \( m \geq 1 \), at \( m^{th} \) order, we have

\[ y(x) = \sum_{m=0}^{M} y_m(x), \] \( \text{when } M \to \infty \text{ we get an accurate approximation of the original Eq (2.2).} \)

Yabushita et. al [28] and Mohamed S. Mohamed et. al [29-31] applied the homotopy analysis method to nonlinear ODE's and suggested the so called optimization method to find out the optimal convergence control parameters by minimum of the square residual error integrated in the whole region having physical meaning. Their approach is based on the square residual error. Let \( \Delta(h) \) denote the square residual error of the governing equation (2.2) and express as

\[ \Delta(h) = \int_{\Omega} (N[u_m(\tau)])^2 d\Omega, \tag{2.15} \]

Where

\[ u_m(\tau) = u_0(\tau) + \sum_{k=1}^{m} u_k(\tau) \tag{2.16} \]

the optimal value of \( h \) is given by a nonlinear algebraic equation as:

\[ \frac{d \Delta(h)}{dh} = 0. \tag{2.17} \]
3 Solving integral equations by the optimal homotopy analysis transform method (OHATM)

In this section, we shall illustrate the optimal homotopy analysis transform technique. To demonstrate the effectiveness of the OHATM algorithm discussed above, several examples of variational problems will be studied in this section. Here all the results are calculated by using the symbolic calculus software Mathematica 7.

Example 3.1. Let us Consider the Abel integral equation:

$$y(x) = \frac{1}{\sqrt{x}} + \pi - \int_0^x \frac{y(t)}{\sqrt{x-t}} dt, \quad 0 \leq x \leq 1,$$

(3.18)

with the initial condition

$$y(x, 0) = \frac{1}{\sqrt{x}} + \pi.$$

(3.19)

with the exact solution

$$y(x) = \frac{1}{\sqrt{x}}.$$

(3.20)

where

$$\mathcal{L}[\varphi(x; q)] = L[\varphi(x; q)],$$

(3.21)

with the property that

$$\mathcal{L}[c] = 0, c \text{ is constant},$$

which implies that

$$\mathcal{L}^{-1}(\ast) = \int_0^\ast(\ast)dt.$$

Taking Laplace transform of equation (3.18) both of sides subject to the initial condition, we get

$$\mathcal{L}[y(x)] - L[\frac{1}{\sqrt{x}} + \pi] + \frac{\pi}{s}(\mathcal{L}[y(x)]) = 0.$$

(3.22)

We now define the nonlinear operator as

$$N[\varphi(x; q)] = L[\varphi(x; q)] - L[\frac{1}{\sqrt{x}} + \pi] + \frac{\pi}{s}L[\varphi(x; q)],$$

(3.23)

and then the $m$th - order deformation equation is given by

$$\mathcal{L}[y_m(x)] - \chi_m y_{m-1}(x) = \hbar H(x)R_m(y_{m-1}).$$

(3.24)

Taking inverse Laplace transform of Eq. (3.24), we get

$$y_m(x) = \chi_m y_{m-1} + \hbar \mathcal{L}^{-1}[H(x)R_m(y_{m-1})],$$

(3.25)

where

$$R_m(y_{m-1}) = \mathcal{L}[y_{m-1}] - L\left[\frac{1}{\sqrt{x}} + \pi\right](1 - \chi_m) + \frac{\pi}{s}(\mathcal{L}[y_{m-1}]),$$

(3.26)

with assumption $H(x) = 1$.

Let us take the initial approximation as

$$y_0(x) = \frac{1}{\sqrt{x}} + \pi,$$

(3.27)

the other components are given by

$$y_1(x) = h\pi + 2h\sqrt{x}\pi,$$

$$y_2(x) = h(1 + h)\pi + 2h\sqrt{x}(h + (1 + h))\pi + h^2x\pi^2.$$
\[ y_3(x) = h(1 + h)^2 \pi + 2h(1 + h)\sqrt{x(2h + (1 + h))}\pi + \frac{4}{3} h^3 x^{\frac{5}{2}} \pi^2 + x \pi^2 (h^3 + 2h^2(1 + h)), \] 

(3.28)

Proceeding in this manner, the rest of the components \( y_n(x) \) for \( n \geq 5 \) can be completely obtained and the series solutions are thus entirely determined. The solution of the problem is given as:

\[ y(x) = y_0(x) + \sum_{m=1}^{\infty} y_m(x), \] 

(3.29)

however, mostly, the results given by the Laplace decomposition method and homotopy perturbation transform method converge to the corresponding numerical solutions in a rather small region. But, different from those two methods, the homotopy analysis transform method provides us with a simple way to adjust and control the convergence region of solution series by choosing a proper value for the auxiliary \( h \) if we select \( h = -1 \), then

\[ y(x) = y_0(x) + \sum_{m=1}^{\infty} y_m(x) = \frac{1}{\sqrt{x}}. \] 

(3.30)

The above result is in complete agreement with [32].

![Figure 1: The comparison between the exact solution and the approximate solution of the Abel integral equation (3.28) at \( h_{optimal} = -0.95 \)]

![Figure 2: The absolute error between the exact solution and the approximate solution of the Abel integral equation (3.28)]
Figure 3: The h-curve of the 4th order approximate solution (3.29)

Note that the solution series contains the auxiliary parameter $h$ which provides us with a simple way to adjust and control the convergence of the solution series. As pointed by Liao,[13] the valid region of $h$ is a horizontal line segment. Therefore, it is straightforward to choose an appropriate range for $h$ that ensures the convergence of the solution series. We stretch the h-curve of $y(0.11)$ in Fig. 3, which shows that the solution series is convergent when $-1 < h < -0.5$.

From Fig. 1 to Fig. 3 shows the graphical comparison between the exact solution and the approximate solution obtained by the OHATM. It can be seen that the solution obtained by the present method nearly identical to the exact solution. The above result is in complete agreement with [32].

Example 3.2. Consider the Abel integral equation:

$$f_1(s) = g_1(s) + \int_0^s (s-t)^3 f_1(t) dt + \int_0^s (s-t)^2 f_2(t) dt,$$

$$f_2(s) = g_2(s) + \int_0^s (s-t)^4 f_1(t) dt + \int_0^s (s-t)^3 f_2(t) dt$$

(3.31)

where

$$g_1(x) = 1 + x^2 - \frac{x^3}{3} - \frac{x^4}{4}, \quad g_2(x) = 1 + x - x^3 - \frac{x^4}{4} - \frac{x^5}{4} - \frac{x^7}{420}$$

(3.32)

with the initial condition,

$$u_0 = 1 + x^2 - \frac{x^3}{3} - \frac{x^4}{4}, \quad v_0 = 1 + x - x^3 - \frac{x^4}{4} - \frac{x^5}{4} - \frac{x^7}{420}$$

(3.33)

with the exact solution

$$f_1(x) = 1 + x^2, \quad f_1(x) = 1 + x - x^3$$

(3.34)

To solve equation (3.31) by means of the homotopy analysis transform method we consider the following linear where

$$L[\varphi(x; q)] = L[\varphi(x; q)],$$

(3.35)

with the property that

$$L[c] = 0, \quad c \text{ is constants},$$

which implies that

$$L^{-1}(\bullet) = \int_0^1 (\bullet) dt.$$

Taking Laplace transform of equation (3.31) both of sides subject to the initial condition, we get

$$L[u(x)] - L \left[ 1 + x^2 - \frac{x^3}{3} - \frac{x^4}{4} \right] - \frac{3!}{s^4} (L[u(x)]) - \frac{2}{s^3} (L[v(x)]) = 0$$

$$L[v(x)] - L \left[ 1 + x - x^3 - \frac{x^4}{4} - \frac{x^5}{420} \right] - \frac{4!}{s^5} (L[u(x)]) - \frac{3!}{s^4} (L[v(x)]) = 0$$

(3.36)
We now define the nonlinear operator as:

\[ N[\varphi_1(x; q), \varphi_2(x; q)] = L[\varphi_1(x; q)] - L \left[ 1 + x^2 - \frac{x^3}{3} - \frac{x^4}{4} \right] - \frac{6}{s^4} L[\varphi_1(x; q)] - \frac{2}{s^3} L[\varphi_2(x; q)] = 0 \]

\[ N[\phi_1(x; q), \phi_2(x; q)] = L[\phi_1(x; q)] - L \left[ 1 + x^2 - x^3 - \frac{x^4}{4} - \frac{x^5}{4} \right] - \frac{24}{s^4} L[\phi_1(x; q)] - \frac{6}{s^3} L[\phi_2(x; q)] = 0 \]

and then the \( m \)th-order deformation equation is given by

\[ L[u_m(x) - \chi_m u_{m-1}(x)] = h_1 H_1(x) R_{1m}(\vec{u}_{m-1}). \]

\[ L[v_m(x) - x_m v_{m-1}(X)] = h_2 H_2(X) R_{2m}(\vec{v}_{m-1}). \]  

(3.37)

Taking inverse Laplace transform of Eq. (3.38), we get

\[ u_m(x, t) = \chi_m u_{m-1} + hL^{-1}[H(x) R_{1m}(\vec{u}_{m-1})], \]

\[ v_m(x, t) = \chi_m v_{m-1} + hL^{-1}[H(x) R_{2m}(\vec{v}_{m-1})], \]

(3.39)

where

\[ R_{1m}(\vec{u}_{m-1}) = L[u_{m-1}] - L \left[ 1 + x^2 - \frac{x^3}{3} - \frac{x^4}{4} \right] \left( 1 - \chi_m \right) - \frac{6}{s^4} (L[u_{m-1}]) - \frac{2}{s^3} (L[v_{m-1}]) = 0, \]

\[ R_{2m}(\vec{v}_{m-1}) = L[v_{m-1}] - L \left[ 1 + x - x^2 - \frac{x^3}{4} - \frac{x^4}{4} - \frac{x^5}{420} \right] \left( 1 - \chi_m \right) - \frac{24}{s^4} (L[u_{m-1}]) - \frac{6}{s^3} (L[v_{m-1}]) = 0, \]

(3.40)

with assumption \( H(x) = 1 \). Let us take the initial approximation as

\[ u_0(x) = 1 + x^2 - \frac{x^3}{3} - \frac{x^4}{4}, \]

\[ v_0(x) = 1 + x - x^3 - \frac{x^4}{4} - \frac{x^5}{4} - \frac{x^7}{420} \]

(3.41)

the other components are given by

\[ u_{11}(x) = -\frac{x^3}{3} + \frac{x^4}{6} + \frac{x^6}{30} + \frac{x^8}{1680} + \frac{x^{10}}{151200}, \]

\[ v_{11}(x) = -\frac{x^2}{4} + \frac{x^3}{20} + \frac{x^5}{60} - \frac{x^7}{3360} + \frac{x^8}{10080} + \frac{x^9}{554400} \]

(3.42)

Hence the solution of the Eq. (3.31) is given as \( A \times h = -1 \) the solution is given by

\[ \Phi_n(x) = \sum_{i=0}^{n-1} u_i(x), \quad n = 1, 2, \ldots \]

\[ u(x) = \lim_{n \to \infty} \Phi_n(x) = \lim_{n \to \infty} \sum_{i=0}^{n-1} u_i(x) \equiv 1 + x^2, \]

(3.43)

\[ \Phi_n(x) = \sum_{i=0}^{n-1} v_i(x), \quad n = 1, 2, \ldots \]

\[ v(x) = \lim_{n \to \infty} \Phi_n(x) = \lim_{n \to \infty} \sum_{i=0}^{n-1} v_i(x) \equiv 1 + x - x^3. \]  

(3.44)

The above result is in complete agreement with \[33\].
The proposed method successfully in putting the new approach to get the optimal value of convergence-controller parameter $\hat{h}$. From the given numerical examples, and Figs 1-4, we conclude that the method is accurate and easy to implement for solving integral equations especially of the second kind.

5 Conclusion

The main aim of this work is to provide the series solution of the integral equation by using the new optimal homotopy analysis transform method (OHATM). Homotopy analysis transform method is coupling of homotopy analysis and Laplace transforms method. The new modification is a powerful tool to search for solutions of integral equation. An excellent agreement is achieved. The proposed method is employed without using linearization, discretization or transformation. It may be concluded that the OHATM is very powerful and efficient in finding the analytical solutions for a wide class of differential and integral equation. The results show that the method is powerful and efficient techniques in finding exact and approximate solutions for integral equations and also, the results show that the OHATM can be a reasonable method to solve this type of integral equations. Also, this method uses simple computation with acceptable solution.

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