A Generalized FDM for solving the Poisson’s Equation on 3D Irregular Domains

J. Izadian\textsuperscript{1*}, M. Jalili\textsuperscript{2}

\textsuperscript{(1)} Department of Mathematics, Faculty of Sciences, Mashhad Branch, Islamic Azad University, Mashhad, Iran
\textsuperscript{(2)} Department of Mathematics, Neyshabur Branch, Islamic Azad University, Neyshabur, Iran

Abstract

In this paper a new method for solving the Poisson’s equation with Dirichlet conditions on irregular domains is presented. For this purpose a generalized finite differences method is applied for numerical differentiation on irregular meshes. Three examples on cylindrical and spherical domains are considered. The numerical results are compared with analytical solution. These results show the performance and efficiency of the proposed method.

Keywords: Poisson’s equation, Finite Difference Method, Generalized Finite Difference Method, Finite Element Method, Dirichlet conditions.


1 Introduction

Numerical solution of Partial Differential Equations (PDE’s) is an important branch of applied mathematics, which has the great application in sciences and engineering. Since the beginning decades of 20\textsuperscript{th} century, various numerical methods have been invented to solve the elliptic PDE’s, in particular the Poisson’s equation. The applications of Finite Difference Method (FDM) for PDE’s have been known since 1900. In 1960, Finite Element Method (FEM) were applied for numerical solution of Ordinary Differential Equations (ODE’s) and PDE’s. The FDM and FEM, a mesh based method, is more suitable when the mesh is regular. Jensen [1] employed FDM with fully arbitrary meshes. Perrone and Kaos [2] formulated a two dimensional FDM capable of using irregular meshes. Liszka and Orkisz [3, 4] developed the Generalized Finite Difference Method (GFDM) which yields better results, when compared to FDM, for uniform node distribution. In recent three decades the spectral methods found a versatile role to solve PDE’s [5, 6, 7]. The applications of these methods for solving Poisson’s equations on irregular domains

* Corresponding Author. Email address: Jalal_Izadian@yahoo.com, Tel: +985118430678
have been reserved to FEM, but that is relatively time-consuming. Recently the various techniques similar to FDM have been proposed to solve PDE’s on irregular domains. Ames [8] presented the principal idea of using FDM to solve PDE’s in irregular domains. Bueno-Orovio et al. [9] applied the spectral method to solve reaction-diffusion equation on irregular 2D domains.

Li and Liu [10] proposed the main objective of the GFDM method which is to approximate the spatial derivatives for a differentiable function in terms of its values at some randomly distributed nodes. GFDM is often used in meshless methods which is suitable for any geometry of the domain. Priete et al. [11] used GFDM to solve an advection–diffusion equation on a cloud like mesh. Benito et al. [12] investigated the effects of weighting function for time dependent problems in GFDM. Gavete et al. [13] Improvement GFDM. Benito et al. [14] purposed an h-adaptive method in GFDM to avoid ill-condition. Then, Benito et al. [15] employed GFDM to solve parabolic and hyperbolic equations. Poplau et al. [16] applied a five points FDM to solve the Poisson’s equation, for studying charge of particles in an accelerator. Izadian et al. [17] have proposed a five points GFDM to solve the Poisson’s equation on irregular 2D domains. The aim of this paper is to generalize GFDM for solving the Poisson’s equation on 3D irregular domains.

2 Description of method

Consider a 3D Poisson’s equation on a 3D domain \( \Omega \), given as follows:

\[
\begin{align*}
    \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} &= f, & \text{On } \Omega, \\
    \text{boundary conditions} \\
    u(x, y, z) &= g(x, y, z), & (x, y, z) \in \partial \Omega,
\end{align*}
\]

(1)

boundary conditions

\[
    u(x, y, z) = g(x, y, z), & (x, y, z) \in \partial \Omega,
\]

(2)

Where \( \Omega \subseteq \mathbb{R}^3 \) is a closed domain in \( \mathbb{R}^3 \), not necessarily a cuboid, \( \partial \Omega \) is the boundary of \( \Omega \), \( f \) and \( g \) are the given functions defined on \( \Omega \) and \( \partial \Omega \), respectively.

Consider the following set of points

\[
\Omega_b = \{(x_i, y_i, z_i) \mid (x_i, y_i, z_i) \in \Omega, \quad i \in K\}, \quad K = \{1, 2, ..., N\},
\]

And

\[
\partial \Omega_b = \{(x_i, y_i, z_i) \in \partial \Omega \mid i \in K_b\}, \quad K_b = \{1, 2, ..., N_b\},
\]

Or

\[
\Omega_i = \{(x_i, y_i, z_i) \in \Omega \mid i \in K_i\}, \quad K_i = \{1, 2, ..., N_i\}.
\]

Then \( N = N_i + N_b \), and \( K = K_i \cup K_b \). By using generalization of partial derivatives introduced in [16,17], to solve (1) and (2) the partial derivatives are approximated as follows:

\[
D_{k,j}^{(l)}u = \sum_{m=1}^{7} \alpha_{l,m}^{(i,k)} u_{k,m}, \quad l = 1, 2, 3, \quad k \in K_i,
\]

(3)

where \( u_{k,m} = u(x_i^{(m)}, y_i^{(m)}, z_i^{(m)}) \), and \( D_{k,j}^{(l)}u \) is \( l^{th} \) order numerical partial derivatives of \( u \) with respect to \( i^{th} \) direction in point \( (x_i, y_i, z_i) \in \Omega_b \). \( \alpha_{l,m}^{(i,k)} \) is a real coefficient. \((x_i^{(m)}, y_i^{(m)}, z_i^{(m)}) \) for \( m = 2, 3, ..., 7 \), are
six distinct mesh points adjacent to \((x_k,y_k,z_k) \in \Omega_h\), with \((x^{(l)}_k,y^{(l)}_k,z^{(l)}_k) = (x_k,y_k,z_k)\). To determine the coefficients \(\alpha^{(l,m)}_{k,j}\), the Taylor expansion with center \((x^{(l)}_k,y^{(l)}_k,z^{(l)}_k)\) of \(u\) in points \((x^{(m)}_k,y^{(m)}_k,z^{(m)}_k)\), \(m = 2,3,\ldots, 7\) is applied. By negligence the terms of higher degrees, and preserving the terms of lower degrees up to two, in (3), for fixed \(l, i\) and \(k\) a linear system of 7 equations, with 7 unknowns \(\alpha^{(l,m)}_{k,j}\) for \(m = 1,2,\ldots, 7\) is obtained. By solving these linear systems, all coefficients \(\alpha^{(l,m)}_{k,j}\) can be determined. These simple system of linear equations can be solved by using the Cramer rule (for more details see [17]). The correct selection of the set of 7 points is essential for well conditioning of these simple linear systems. The coefficient matrix is normally well conditioned if the set of selected points are not in the same plane. Numerical experiences show that it is preferable that at least two points of 7 points molecule situate on two different horizontal planes, above and below \((x^{(l)}_k,y^{(l)}_k,z^{(l)}_k)\). After computing all coefficient \(\alpha^{(l,m)}_{k,j}\), for the internal points of \(\Omega_h\), by replacing in (1), the following discretized equations are obtained:

\[
D_{k,1}^{(2)} u + D_{k,2}^{(2)} u + D_{k,3}^{(2)} u = f(x_k, y_k, z_k), \quad k \in K_i, \tag{4}
\]

Or

\[
\sum_{m=1}^{7} \alpha^{(2,m)}_{k,m} u_{k,m} + \sum_{m=1}^{7} \alpha^{(2,m)}_{2,k} u_{k,m} + \sum_{m=1}^{7} \alpha^{(2,m)}_{3,k} u_{k,m} = f_k, \quad k \in K_i, \tag{5}
\]

where \(f_k = f(x_k, y_k, z_k)\). Equation (5) constitute a linear system of \(N_i\) equations, with \(N_i\) unknowns. Thus by accepting the notations

\[
V = (u_1, u_2, \ldots, u_{N_i}),
\]

\[
g_k = g(x_k, y_k, z_k), \quad k \in K_h.
\]

The linear systems of equations (5) can be simplified as:

\[
AV = b, \tag{6}
\]

where \(A\) is a \(N_i \times N_i\) matrix which its elements are certain linear combination of some coefficients \(\alpha^{(l,m)}_{i,k}\) and \(b\) is a column matrix with some \(f_k\) or the terms that contain values \(g_k\) multiplied by certain \(\alpha^{(l,m)}_{i,k}\) adding with some \(f_k\). If one uses the natural enumeration, and a will be a sparse band matrix. By solving (6), the vector \(V\) which contains, the approximate values of solution on internal mesh points, can be determined.

3 Numerical experiments

In this section the proposed generalized finite difference method, is applied to spherical domains and a domain limited to two cylinders, with the same axis, and two planes parallel with \(xOy\) plane. The numerical and analytical solution, are compared.

Example 1.

Consider the domain \(\Omega\) in equation (1), is given by

\[
\Omega_h = \left\{ (x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1 \right\},
\]
and $f(x, y, z) = 6$, $g(x, y, z) = x^2 + y^2 + z^2$, $N = 665$, and $N_i = 429$. Certain results on mesh points and absolute error are given in Table 1. $u^*$ is approximate of solution $u$ in the given point. Figure 1 presents the approximate surface of solution for a fixed value of $z$. Noting that the exact solution is $u(x, y, z) = x^2 + y^2 + z^2$.

Table 1: Some numerical results of example 1

<table>
<thead>
<tr>
<th>$i$</th>
<th>$(x_i, y_i, z_i)$</th>
<th>$u^*(x_i, y_i, z_i)$</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-0.2,-0.2,-0.8)</td>
<td>0.6784</td>
<td>-0.0116$\times10^{-8}$</td>
</tr>
<tr>
<td>2</td>
<td>(-0.6,-0.4,-0.6)</td>
<td>0.5424</td>
<td>-0.0022$\times10^{-8}$</td>
</tr>
<tr>
<td>3</td>
<td>(-0.6,0.0,-0.2)</td>
<td>0.1120</td>
<td>0.1620$\times10^{-8}$</td>
</tr>
<tr>
<td>4</td>
<td>(-0.2,-0.8,0.4)</td>
<td>0.4288</td>
<td>0.0087$\times10^{-8}$</td>
</tr>
<tr>
<td>5</td>
<td>(-0.4,0.2,0.8)</td>
<td>0.6976</td>
<td>0.0246$\times10^{-8}$</td>
</tr>
<tr>
<td>6</td>
<td>(0.4,-0.4,0.8)</td>
<td>0.8192</td>
<td>-0.0757$\times10^{-8}$</td>
</tr>
</tbody>
</table>

Figure 1: The approximate surface of solution for a fixed $z$

**Example 2.**

Consider $\Omega$ in equation (1), is given as follows

$$\Omega_h = \left\{(x, y, z) \in \mathbb{R}^3 \mid 0.25 \leq x^2 + y^2 \leq 1, -1 \leq z \leq 1\right\},$$

And $f(x, y, z) = 6$, $g(x, y, z) = x^2 + y^2 + z^2$ and exact solution is $u(x, y, z) = x^2 + y^2 + z^2$. By considering a mesh of points with $N = 576$, and $N_i = 225$. The numerical solution is computed. Some results and absolute error are given in Table 2. $u^*$ is approximate of solution $u$ in the given point. The approximate surface of solution for fixed $z$ is presented in Fig 2.

Table 2: Some numerical results of example 2

<table>
<thead>
<tr>
<th>$i$</th>
<th>$(x_i, y_i, z_i)$</th>
<th>$u^*(x_i, y_i, z_i)$</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-0.5,-0.75,0.75)</td>
<td>1.1250</td>
<td>-0.0025$\times10^{-13}$</td>
</tr>
<tr>
<td>2</td>
<td>(0.65,-0.5,0.5)</td>
<td>0.4810</td>
<td>-0.0420$\times10^{-13}$</td>
</tr>
<tr>
<td>3</td>
<td>(0.25,-0.75,0)</td>
<td>0.6090</td>
<td>0.0050$\times10^{-13}$</td>
</tr>
<tr>
<td>4</td>
<td>(-0.5,-0.75,0.5)</td>
<td>0.6250</td>
<td>0.0058$\times10^{-13}$</td>
</tr>
<tr>
<td>5</td>
<td>(0.5,0.75,0.75)</td>
<td>1.1250</td>
<td>-0.0010$\times10^{-13}$</td>
</tr>
<tr>
<td>6</td>
<td>(0.5,0.5,1)</td>
<td>0.3750</td>
<td>0</td>
</tr>
</tbody>
</table>
Example 3.

Consider $\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid 0.25 \leq x^2 + y^2 \leq 1, \quad -1 \leq z \leq 1\},$

and $f(x, y, z) = -2, \quad g(x, y, z) = x^2 - y^2 - z^2$ and exact solution is $u(x, y, z) = x^2 - y^2 - z^2$. By considering a mesh of points with $N = 576$, and $N_i = 225$. Some numerical results and absolute error are given in Table 3. $u^*$ is approximate of solution $u$ in the given. The approximate surface of solution for fixed $z$ are presented in Fig 3.

<table>
<thead>
<tr>
<th>$i$</th>
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<th>$u^*(x_i, y_i, z_i)$</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-0.25,-0.5,-0.75)</td>
<td>0.8750</td>
<td>-0.0022×10^{13}</td>
</tr>
<tr>
<td>2</td>
<td>(0.75,-0.5,-0.5)</td>
<td>0.8125</td>
<td>-0.0450×10^{13}</td>
</tr>
<tr>
<td>3</td>
<td>(0,-0.75,0)</td>
<td>0.5625</td>
<td>0.0055×10^{13}</td>
</tr>
<tr>
<td>4</td>
<td>(-0.5,0,0.5)</td>
<td>0.5000</td>
<td>0.0042×10^{13}</td>
</tr>
<tr>
<td>5</td>
<td>(0.5,0.75,0.75)</td>
<td>1.3750</td>
<td>-0.0011×10^{13}</td>
</tr>
<tr>
<td>6</td>
<td>(0.65,0.5,1)</td>
<td>1.6725</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 3: The approximate surface of solution for a fixed $z$

4 Conclusion and Discussion

The proposed GFDM, is applied to obtain numerical solution of the Poisson’s equation on irregular domains, this numerical method is more complicated than standard FDM on cell domains. In particular the
The coefficient matrix of the resulting linear system for determining the coefficients of derivatives is very sensible to 7 point molecules and can be ill-conditioned. But the possibility of choosing arbitrarily non-regular meshes, in this method, allows it to be a suitable tool to reconstruct the mesh and solve the problem. Thus the proposed method, using FDM and GFDM, is very suitable for solving PDE’s in irregular domains. These numerical results for two irregular domains demonstrate the reliability and the performance of the proposed method.

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