A Numerical Method For Solving Nth-Order Fuzzy Differential Equation by using Characterization Theorem

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Abstract
In this paper, we study a numerical method for solving Nth-Order fuzzy differential equation (NOFDE). A Characterization Theorem that is presented shows that the NOFDE is equivalent to system of ODEs. So the numerical method for ODE can be used for NOFDE. Two examples are provided.

Keywords: Nth-order Fuzzy differential equations; Numerical method; Characterization Theorem

1 Introduction
Fuzzy differential equations (FDEs) are used in modeling problems of science and engineering. Most of the science and engineering applications of FDEs require the solution of an FDE subject to some fuzzy initial conditions, therefore, a fuzzy initial value problem arises. It is too complicated to obtain the exact solutions of an FDE that arises in the real applications. Since the fuzzy derivative is used in FDE, it is natural to begin by presenting a background of fuzzy derivative. The first and the most popular approach is using the Hukuhara differentiability, or the Seikkala derivative for fuzzy valued functions. First-order fuzzy differential equations have been considered, for example, in [16, 21, 22, 5]. In the last few years, many works have been performed by several authors in numerical solutions of fuzzy differential equations [1, 2, 18, 4]. Recently, the numerical solutions

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of fuzzy differential equations by predictor-corrector method has been studied in [3]. In [19, 20], two-point boundary value problems relative to second-order nonlinear fuzzy differential equations are studied, providing results on existence and uniqueness of solution. Under this setting, mainly the existence and uniqueness of solutions is studied. In [7] Bede proved a characterization theorem which states that under certain conditions a fuzzy differential equation is equivalent to a system of ordinary differential equations. Bede also remarked that this characterization theorem can help to numerically solve fuzzy differential equation by converting them to systems of ODEs which can then be solved by any suitable numerical method for ODEs. More specifically, in [7] Bede wrote, in order to obtain numerical solutions of fuzzy differential equations under Hukuhara differentiability, it is not necessary to rewrite the whole literature on numerical solutions of ODEs in the fuzzy setting, but instead we can use any numerical method directly. In this paper we will study solutions to

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y^{(1)} + a_0(x)y = g(x),$$

where the $a_i(x), 0 \leq i \leq (n-1)$, and $g(x)$ are continuous on some interval $I$, subject to initial conditions

$$y(0) = \bar{k}_0, y^{(1)}(0) = \bar{k}_1, \ldots, y^{(n-1)}(0) = \bar{k}_{(n-1)},$$

for fuzzy numbers $\bar{k}_i, 0 \leq i \leq (n-1)$. The interval $I$ can be $[0, T]$ for some $T > 0$ or $I = [0, \infty)$. This problem is called the fuzzy initial value problem for linear differential equations. This paper is organized as follows: In Section 2, we provide some background on fuzzy differential equation and in Section 3 we review the Nth-Order fuzzy differential equation IVPs NOFDE, we proved existence and uniqueness solution for NOFDE by using Characterization Theorem. Two example are presented in Section 4 and the conclusions are given in the final Section 5.

2 Preliminaries

**Definition 2.1.** Let $K_F(R^n)$ denote the family of all non-empty, compact, convex subsets of $R^n$. Denote by $E^n$ the set of $\bar{u} : R^m \to [0, 1]$ such that $\bar{u}$ satisfies (i) – (iv) mentioned next:

1. $\bar{u}$ is normal that is, there exists an $y_0 \in R^m$ such that $\tilde{u}(y_0) = 1$,
2. $\bar{u}$ is fuzzy convex,
3. $\bar{u}$ is upper semi continuous,
4. $[\bar{u}]^0 = \{y \in R^m : \bar{u}(y) > 0\}$ is compact.

We denote the alpha-level set $[\tilde{u}]^\alpha = \{y \in R^m : \bar{u}(y) \geq \alpha\}$ for $0 < \alpha \leq 1$. Clearly the alpha-level sets $[\bar{u}]^\alpha \in K_F(R^m)$.

The notation

$$[\tilde{y}]^\alpha = [\tilde{y}^\alpha, \bar{y}^\alpha] \quad t \in I, \quad 0 < \alpha \leq 1.$$ 

denotes explicitly the alpha-level set of $y$. We refer to $\tilde{y}$ and $\bar{y}$ as the lower and upper branches on $y$, respectively. For $y \in K_F(R^m)$, we define the length of $y$ as:

$$\text{len}(y) = \bar{y} - \tilde{y}.$$ 

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The following remark shows when \([\underline{y}^\alpha, \overline{y}^\alpha]\) is a valid \(\alpha\)-level set of a fuzzy real number.

**Remark 2.1.** [15] The sufficient and necessary conditions for \([\underline{y}^\alpha, \overline{y}^\alpha]\) to define the parametric form of a fuzzy number are as follows:

- (i) \(\underline{y}^\alpha\) is a bounded monotonic increasing (nondecreasing) left-continuous function \(\forall \in [0, 1]\) and right-continuous for \(\alpha \in 0\).

- (ii) \(\overline{y}^\alpha\) is a bounded monotonic decreasing (nonincreasing) left-continuous function \(\forall \in [0, 1]\) and right-continuous for \(\alpha \in 0\).

- \(\underline{y}^\alpha \leq \overline{y}^\alpha, 0 \leq \alpha \leq 1\).

Let \(\tilde{x}, \tilde{y} \in E^m\). If there exists \(\tilde{z} \in E^m\) such that \(\tilde{x} = \tilde{y} + \tilde{z}\), then \(\tilde{z}\) is called the Hukuhara difference of \(\tilde{x}\) and \(\tilde{y}\) and it is denoted by \(\tilde{x} \oplus \tilde{y}\). In this paper the \(\oplus\) sign stands always for Hukuhara difference and let us remark that

\[
\tilde{x} \oplus \tilde{y} \neq \tilde{x} + (-1)\tilde{y}.
\]

In the space \(E^m\), it is possible to define an addition and a multiplication for a real number and the defined distance \(d\) verifies that

\[
d(u + w, v + w) = d(u, v), u, v, w \in E^m,
\]

\[
d(\lambda u, \lambda v) = \lambda d(u, v), u, v \in E^m, \lambda > 0,
\]

\[
d(u + w, v + z) \leq d(u, v) + d(w, z), u, v, w, z \in E^m.
\]

**Definition 2.2.** Let \(d_H(A, B)\) be the Hausdorff distance between sets \(A, B \in E^m\)

\[
d(\tilde{A}, \tilde{B}) = \sup\{d_H([\tilde{A}]^\alpha, [\tilde{B}]^\alpha) : \alpha \in [0, 1]\}
\]

and \((E^m, d)\) is a complete metric space, for more details see [12].

**Definition 2.3.** Let \(\tilde{f} : [t_0, T] \rightarrow E^m\) and \(y_0 \in [t_0, T]\). We say that \(\tilde{f}\) is Hukuhara differentiable at \(y_0\), if there exists an element \(\tilde{f}' \in E^m\) such that for all \(h > 0\) sufficiently near to 0, exist \(\tilde{f}(y_0 + h) \ominus \tilde{f}(y_0), \tilde{f}(y_0) \ominus \tilde{f}(y_0 - h)\) and the limits are taken in the metric space \((E^m, D)\)

\[
\lim_{h \rightarrow 0^+} \left( \frac{\tilde{f}(y_0 + h) \ominus \tilde{f}(y_0)}{h} \right) = \lim_{h \rightarrow 0^+} \left( \frac{\tilde{f}(y_0) \ominus \tilde{f}(y_0 - h)}{h} \right) = \tilde{f}'(y_0).
\]

The fuzzy set \(\tilde{f}'(y_0)\) is called the Hukuhara derivative of \(\tilde{f}\) at \(y_0\).

These limits are taken in the space \((E^m, d)\) if \(t_0\) or \(T\), then we consider the corresponding One-Side derivation. Recall that \(\tilde{U} \ominus \tilde{V} = \tilde{W} \in E^m\) are defined on \(\alpha\)-level sets, where \([\tilde{U}]^\alpha \ominus [\tilde{V}]^\alpha = [\tilde{W}]^\alpha\) for all \(\alpha \in [0, 1]\). By consideration of definition of the metric \(d\) all the \(\alpha\)-level set \([\tilde{f}(0)]^\alpha\) are Hukuhara differentiable at \(y_0\) with Hukuhara derivatives \([\tilde{f}'(y_0)]^\alpha\) for each \(\alpha \in [0, 1]\), when \(\tilde{f} : [t_0, T] \rightarrow E^m\) is Hukuhara differentiable at \(y_0\) with Hukuhara derivative \(\tilde{f}'(y_0)\).
Remark 2.2. [12] The integral of $f$ over $[t_0, T]$, denoted by $\int_{[t_0, T]} f(t)dt$ or $\int_{t_0}^{T} f(t)dt$, is defined levelwise by

$$[\text{int}_{[t_0, T]} f(t) dt]_a = \int_{[t_0, T]} f_\alpha(t) dt = \int_{t_0}^{T} f_\alpha(t) dt = \{ \int_{[t_0, T]} g(t) dt : g : [t_0, T] \to R^m \text{ is a measurable selection for } f_\alpha \}$ for $\alpha \in (0, 1)$.

We say that $f$ is integrable over $[t_0, T]$ if $\int_{[t_0, T]} f(t) dt \in E^m$. The continuity of $f : [t_0, T] \to E^m$ provides the integrability of $f$ and for $f, g$ integrable functions, $d(f, g)$ is integrable and $d(\int f, \int g) \leq d(f, g)$.

2.1 Nth-order linear differential equations with fuzzy initial conditions

In section we review Nth-Order fuzzy initial value problems, that it has been studied in [12].

$$\begin{align*}
\begin{cases}
x^{(n)}(t) = f(t, x(t), x^{(1)}(t), \ldots, x^{(n-1)}(t)), & t \in [t_0, T], \\
x(t_0) = k_1, & x^{(1)}(t_0) = k_2, \ldots, x^{(n-1)}(t_0) = k_n
\end{cases}
\tag{2.2}
\end{align*}$$

where $f : [t_0, T] \times (E^m)^n \to E^m$ continuous and $k_1, k_2, \ldots, k_n$ are real constants and $x^{(i)}$ represents the $i$th-derivative of $x$ in the sense of Hukuhara.

Theorem 2.1. [12] Consider the space

$$C^n(I, E^m) = \{ x \in C(I, E^m) : \exists \ x', \ldots, x^{(n)} \in C(I, E^m) \},$$

durnished with the distance

$$H_n(x, y) = H(x, y) + H(x', y') + \ldots + H(x^{(n)}, y^{(n)}) = \sum_{i=1}^{n} H(x^{(i)}, y^{(i)}),$$

where of course, $x^{(0)} = x$. Then, for every $n \in N, n \geq 0, (C^n(I, E^m), H_n)$ is a complete metric space. We obtain an integral representation for the initial value problem Eq.(2.2) in order to prove the existence of a unique solution using fixed point theory [12].

Theorem 2.2. [12] The function $x \in C^n(I, E^m)$ is a solution to problem Eq.(2.2) if and only if $x$ satisfies the following integral equation for all $z_1 \in [t_0, T]$.

$$\begin{align*}
x(z_1) = k_1 + k_2(z_1 - t_0) &+ k_3 \int_{t_0}^{z_1} (z_2 - t_0) dz_2 + k_4 \int_{t_0}^{z_1} \int_{t_0}^{z_2} (z_3 - t_0) dz_3 dz_2 \\
&+ \ldots + k_n \int_{t_0}^{z_1} \int_{t_0}^{z_2} \ldots \int_{t_0}^{z_{n-2}} (z_{n-1} - t_0) dz_{n-1} \ldots dz_3 dz_2 \\
&+ \int_{t_0}^{z_1} \int_{t_0}^{z_2} \ldots \int_{t_0}^{z_{n-1}} f(s, x(s), \ldots, x^{(n-1)}(s)) ds dz_{n-1} \ldots dz_3 dz_2.
\end{align*}$$

Theorem 2.3. [12] Let $f : [t_0, T] \times (E^m)^n \to E^m$ be continuous, and suppose that there exist $M_1, M_2, \ldots, M_n > 0$ such that

$$d(f(t, x_1, x_2, \ldots, x_n), f(t, y_1, y_2, \ldots, y_n)) \leq \sum_{i=1}^{n} M_i d(x_i, y_i), \tag{2.3}$$

for all $t \in [t_0, T], x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in E^m$. Then the initial value problem Eq.(2.2) has a unique solution on $[t_0, T]$.  

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The definition 2.3 is a straightforward generalization of the Hukuhara differentiability of a set-valued function. So if \( F \) is differentiable at \( t_0 \in [t_0 + a, T] \), then all its \( \alpha \)-levels \( F_\alpha(t) = [F(t)]^\alpha \) are Hukuhara differentiable at \( t_0 \) and \([F'(t_0)]^\alpha = D F_\alpha(t_0)\), where \( DF_\alpha \) denotes the Hukuhara derivative of \( F_\alpha \).

**Theorem 2.4.** [15] Let \( F: (t_0 + a, T) \to K_F(R^m) \) be Hukuhara differentiable and denote \([F(t)]^\alpha = [F_\alpha(t), F^\alpha(t)]\). Then the boundary functions \( F^\alpha(t), F^\alpha(t) \) are differentiable

\[
[F'(t)]^\alpha = [(F^\alpha(t))', (F^\alpha(t))'] \quad \alpha \in [0, 1]
\]

### 3 Theorem for the solutions of NOFDEs by using ODEs

Let us consider the fuzzy initial value problem NOFDE

\[
\begin{cases}
  x^{(n)}(t) = f(t, x(t), x^{(1)}(t), \ldots, x^{(n-1)}(t)), & t \in [t_0, T],
  \\
x(t_0) = k_1, & \quad x^{(1)}(t_0) = k_2, \ldots, x^{(n-1)}(t_0) = k_n
\end{cases}
\]

(3.4)

where \( f: [t_0, T] \times (E^m)^n \to E^m \) continuous and \( k_1, k_2, \ldots, k_n \) are real constants and \( x^{(i)} \) represents the \( i \)-th derivative of \( x \) in the sense of Hukuhara. Eq.(3.4) translates into the following system of ODEs

\[
\begin{cases}
  (x^a(t))^{(n)} = f^a(t, x^a(t), \ldots, (x^a(t))^{(n-1)}), \\
  (x^a(t))' = \overline{f}^a(t, x^a(t), \ldots, (x^a(t))'), \\
  \overline{x^a(t)} = k_1, & \quad \overline{x^a(t)} = k_1,
  \\
  (x^a(t_0))^{(n-1)} = k_2, & \quad (x^a(t_0))' = k_2,
  \\
  \vdots & \quad \vdots
\end{cases}
\]

(3.5)

where

\[
[f(t, x, x', \ldots, x^{(n-1)})]^a = \\
[f^a(t, x^a(t), \ldots, (x^a(t))^{(n-1)}), \\
(\overline{f}^a(t, x^a(t), \ldots, (x^a(t)))', \ldots, (x^a(t))^{(n-1)}, (\overline{x^a(t)})^{(n-1)})]
\]

In the following theorem we show that the NOFDE Eq.(3.4) will be equivalent to system Eq.(3.5). The numerical solutions of the ODEs are extremely well studied in the literature, so any numerical method we can consider for the system of ODEs, since the solution will be as well as solution of the NOFDE.

**Proposition 3.1.** Let \( F: (t_0 + a, T) \to K_F(R^m) \) is Hukuhara differentiable \( n \) times and

\[
[F'(t)]^\alpha = [(F^\alpha(t))', (F^\alpha(t))]' \quad \alpha \in [0, 1]
\]

that \((F^\alpha)'(t), (F^\alpha)'(t)\) are differential. We can write

\[
[F^n(t)]^\alpha = [(F^\alpha(t))^n, (F^\alpha(t))^n] \quad \alpha \in [0, 1].
\]
Let us consider the NOFDE Eq.(3.4) where $y^x$ is differentiable at $t \in E^m$ if there exits a $F^m(t) \in E^m$ such that the limits
\[
\lim_{h \to 0} \left( \frac{F^{n-1}(t+h) \ominus F^{n-1}(t)}{h} \right) = \lim_{h \to 0} \left( \frac{F^{n-1}(t) \ominus F^{n-1}(t-h)}{h} \right)
\]
exist and equal to $F^n(t)$. Now
\[
[F^{n-1}(t+h) \ominus F^{n-1}(t)]^\alpha = [(F^\alpha)^{-1}(t+h) \ominus (F^\alpha)^{-1}(t)]
\]
and similarly for $[F^{n-1}(t) \ominus F^{n-1}(t-h)]^\alpha$. Dividing by $h$ and passing to the limit gives the proposition.

**Theorem 3.1.** Let us consider the NOFDE Eq.(3.4) where $f : [t_0, t_0+a] \times E^m \times E^m \to E^m$ is such that

1. $[f(t, x, x', \ldots, x^{(n-1)})]^\alpha = \int f^\alpha(t, x(t), \bar{y}(t), (x(t))', (\bar{y}(t))', \ldots, (x(t))^{(n-1)}, (\bar{y}(t))^{(n-1)})$
2. There exist $L > 0$ such that
\[
\left| \int f^\alpha(t, x(t), \bar{y}(t), (x(t))', (\bar{y}(t))', \ldots, (x(t))^{(n-1)}, (\bar{y}(t))^{(n-1)}) \right| \leq L \max \{|x-x^1|, |y-y^1|, |x'-y'|, |x^{(n-1)}-x^{(n-1)}| \}
\]
and
\[
\left| \int f^\alpha(t, x(t), \bar{y}(t), (x(t))', (\bar{y}(t))', \ldots, (x(t))^{(n-1)}, (\bar{y}(t))^{(n-1)}) \right| \leq L \max \{|x-x^1|, |y-y^1|, |x'-y'|, |x^{(n-1)}-x^{(n-1)}| \}
\]
3. $f^\alpha$ and $\tilde{f}^\alpha$ are equicontinuous.

Then the NOFDE Eq.(3.4) and the system of ODE Eq.(3.5) are equivalent.

**Proof.** According to definition 2.3, we can defined a mapping $F^\alpha : [t_0, T] \to E^m$ is differentiable at $t \in E^m$ if there exits a $F^m(t) \in E^m$ such that

\[
\sup \max \{|f^\alpha(t, x(t), \bar{y}(t), (x(t))', (\bar{y}(t))', \ldots, (x(t))^{(n-1)}, (\bar{y}(t))^{(n-1)}) - \int f^\alpha(t, x(t), \bar{y}(t), (x(t))', (\bar{y}(t))', \ldots, (x(t))^{(n-1)}, (\bar{y}(t))^{(n-1)})| \}
\]

Further, the Lipschitz property in condition (2), we can show property as follows:

\[
\sup \max \{|f^\alpha(t, x(t), \bar{y}(t), (x(t))', (\bar{y}(t))', \ldots, (x(t))^{(n-1)}, (\bar{y}(t))^{(n-1)}) - \int f^\alpha(t, x(t), \bar{y}(t), (x(t))', (\bar{y}(t))', \ldots, (x(t))^{(n-1)}, (\bar{y}(t))^{(n-1)})| \}
\]

Proof. The equicontinuous $f^\alpha$ and $\tilde{f}^\alpha$ implies the continuity of the function $f$. Further, the Lipschitz property in condition (2), we can show property as follows:

\[
\sup \max \{|f^\alpha(t, x(t), \bar{y}(t), (x(t))', (\bar{y}(t))', \ldots, (x(t))^{(n-1)}, (\bar{y}(t))^{(n-1)}) - \int f^\alpha(t, x(t), \bar{y}(t), (x(t))', (\bar{y}(t))', \ldots, (x(t))^{(n-1)}, (\bar{y}(t))^{(n-1)})| \}
\]
\[ L \max \{ |x - x|, |y - y|, |x' - x'|, |y' - y'|, \ldots, |x^{(n-1)} - x^{(n-1)}|, |y^{(n-1)} - y^{(n-1)}| \}, \]

by the Hausdorff distance \( d_H \) property

\[ d_H(x, y) = \sup \max \{ |x - y|, |x| \}, \]

\[ d_H(x', y') = \sup \max \{ |x' - y'|, |x'| \}, \]

\[ \vdots \]

\[ d_H(x^{(n-1)}, y^{(n-1)}) = \sup \max \{ |x^{(n-1)} - y^{(n-1)}|, |x(n-1) - y(n-1)| \} \]

and by distance \( d \) property

\[ d(u + w, v + z) \leq d(u, v) + d(w, z), \]

consequence

\[ \sup \max \{ |f^\alpha(t, x(t), y(t)), (x(t))', (y(t))', \ldots, (x(t))^{(n-1)}, (y(t))^{(n-1)}| -, \]

\[ f^\alpha(t, x(t), y(t), (x(t))', (y(t))', \ldots, (x(t))^{(n-1)}, (y(t))^{(n-1)}|, \]

\[ |\bar{f}^\alpha(t, x(t), y(t), (x(t))', (y(t))', \ldots, (x(t))^{(n-1)}, (y(t))^{(n-1)}| - \]

\[ \bar{f}^\alpha(t, x(t), y(t), (x(t))', (y(t))', \ldots, (x(t))^{(n-1)}, (y(t))^{(n-1)}| \} \]

\[ \leq L \sup \max \{|x' - y'|\} + L \sup \max \{|x - y|\} + \ldots + L \sup \max \{|x^{(n-1)} - y^{(n-1)}|\} + L \sup \max \{|x^{(n-1)} - y^{(n-1)}|\} \]

finally

\[ \sup \max \{ |f^\alpha(t, x(t), y(t)), (x(t))', (y(t))', \ldots, (x(t))^{(n-1)}, (y(t))^{(n-1)}| - \]

\[ f^\alpha(t, x(t), y(t), (x(t))', (y(t))', \ldots, (x(t))^{(n-1)}, (y(t))^{(n-1)}|, \]

\[ |\bar{f}^\alpha(t, x(t), y(t), (x(t))', (y(t))', \ldots, (x(t))^{(n-1)}, (y(t))^{(n-1)}| - \]

\[ \bar{f}^\alpha(t, x(t), y(t), (x(t))', (y(t))', \ldots, (x(t))^{(n-1)}, (y(t))^{(n-1)}| \} \]

\[ \leq M d(x, y) + M_1 d(x', y') + \ldots + M_{n-1} d(x^{(n-1)}, y^{(n-1)}). \]  \( (3.6) \)

According to Theorem 3.1, it shows NOFDE Eq.(3.4) has a unique solution. By proposition 3.1, we can show that the solution of NOFDE are Hukuhara differentiable and so, implies the functions (\( \pi ^\alpha \)) and (\( \xi ^\alpha \)) are differentiable, and as a conclusion (\( (\pi ^\alpha), (\xi ^\alpha) \)) is a solution of Eq.(3.5). Conversely. In [17], Kaleva states that if we ensure that the solution (\( \pi ^\alpha, \xi ^\alpha \)) of the system first order ODEs are valid level sets of a fuzzy number valued function and if the derivatives (\( (\pi ^\alpha), (\xi ^\alpha) \)) are valid level sets of a fuzzy-valued function, then by using the stacking Theorem we can construct the solution of the FIVP. Now, we can expand this process for system Eq.(3.5) and NOFDE Eq.(3.4), this solution of system Eq.(3.5) exists by Lipschitz condition (2), and the solution is a unique. Let us suppose that we have a solution (\( (\pi ^\alpha)^{(n-1)}, (\xi ^\alpha)^{(n-1)} \)), with \( \alpha \in [0, 1] \) fixed, of the system Eq.(3.5). Also, the Eq.(3.6) implies the existence and uniqueness of the fuzzy solution \( \tilde{x} \). Now, since \( \tilde{x} \) is Hukuhara differentiable, (\( \pi ^\alpha), (\xi ^\alpha) \)) the endpoints of (\( \tilde{x} \)) which are obviously valid level sets of a fuzzy-valued function) is a solution of Eq.(3.5). Since the solution of Eq.(3.5) is unique, we have (\( \tilde{x} \))\( ^\alpha \), that is the problems Eq.(3.4) and Eq.(3.5) are equivalent.
4 Numerical Example

Example 4.1. [8] Consider the circuit shown in Fig. 1 where \(L = 1\ h\), \(R = 2\Omega\), \(C = 0.25 f\) and \(E(t) = 50\cos t\). If \(Q(t)\) is the charge on the capacitor at time \(t > 0\), then

\[
\begin{align*}
Q''(t) + 2Q'(t) + 4Q(t) &= 50 \cos t, \\
\left[Q(0)\right] = [4 + \alpha, 6 - \alpha], & \quad 0 \leq \alpha \leq 1, \\
\left[Q'(0)\right] = [\alpha, 2 - \alpha].
\end{align*}
\] (4.7)

Let

\[
\begin{align*}
\left\{ \begin{array}{l}
\quad u_1(t) = Q(t)^\alpha \\
\quad u_2(t) = Q'(t)^\alpha \\
\end{array} \right.
\quad \begin{array}{l}
u_1'(t) = u_2(t), \\
\quad u_2'(t) = 50 \cos t - 4u_1(t) - 2u_2(t).
\end{array}
\end{align*}
\] (4.8)

The unique solution is

\[
Q^\alpha(t) = [(4 + \alpha)\frac{2e^{-t}}{\sqrt{3}} \sin(\sqrt{3}t + \frac{\pi}{3}) + (\alpha)\frac{e^{-t}}{\sqrt{3}} \sin(\sqrt{3}t) + \Psi(t),
(6 - \alpha)\frac{2e^{-t}}{\sqrt{3}} \sin(\sqrt{3}t + \frac{\pi}{3}) + (2 - \alpha)\frac{e^{-t}}{\sqrt{3}} \sin(\sqrt{3}t) + \Psi(t)]
\]

where

\[
\Psi = -\Delta_0 e^{-t} \cos(\sqrt{3}t) - \frac{\Delta_0 + \Delta_1}{\sqrt{3}} e^{-t} \sin(\sqrt{3}t) + G(t)
\]

with

\[
G(t) = \frac{100}{13} \sin(t) + \frac{150}{13} \cos(t).
\]

Define

\[
\Delta_0 = G(0), \ldots, \Delta_{n-1} = G^{n-1}(0).
\]
By using Runge-Kutta, we present the numerical solution of this example at \( t = 2 \) in Fig. 2.

![Figure 2](image)

**Fig. 2.** The exact solution SOFDE (solid graph) and the approximate solution SOFDE (dashed graph) by using Runge-Kutta method.

**Example 4.2.** [8]. Consider the vibrating mass \((m = 1 \text{ slug})\) in Fig. 3. The spring constant is \(k = 4 \text{ lb ft}^{-1}\), there is no damping force and the forcing function is \(100 \cos \varepsilon t\) for \(\varepsilon > 0\). The differential equation of motion is

\[
\begin{aligned}
\ddot{y}(t) + 4y(t) &= 100 \cos \varepsilon t, \\
[y(0)]^\alpha &= [-1 + \alpha, 1 - \alpha], \\
[y'(0)]^\alpha &= [-1 + \alpha, 1 - \alpha]
\end{aligned}
\]

\[\text{(4.9)}\]

Let

\[
\begin{aligned}
\begin{cases}
  u_1(t) = y(t)^\alpha \\
  u_2(t) = y'(t)^\alpha
\end{cases}
\end{aligned}
\]
\[
\begin{aligned}
\begin{cases}
u'_1(t) &= u_2(t), \\
u'_2(t) &= 100 \cos(\varepsilon t) - 4u_1(t).
\end{cases}
\end{aligned}
\] (4.10)

The unique solution is

\[
y = \left[ (-1 + \alpha) \cos(2t) + \frac{-1 + \alpha}{2} \sin(2t) + \Psi(t),
(1 - \alpha) \cos(2t) + \frac{1 - \alpha}{2} \sin(2t) + \Psi(t) \right]
\]

for

\[
\Psi(t) = \frac{100}{4 - \varepsilon^2} (\cos(\varepsilon t) - \cos(2t)).
\]

By using Runge-Kutta, we present the numerical solution of this example at \( t = 2 \) in Fig.4.

![Fig4. Result of Example 2 for \( t = 2 \) by using Runge-Kutta method](image)

<table>
<thead>
<tr>
<th>( r )</th>
<th>( y_p )</th>
<th>( y_b )</th>
<th>( Exa_p )</th>
<th>( Exa_b )</th>
<th>( Error_p )</th>
<th>( Error_b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-0.587346</td>
<td>1.57145</td>
<td>-0.587561</td>
<td>1.57171</td>
<td>0.000215275</td>
<td>0.000251586</td>
</tr>
<tr>
<td>0.1</td>
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<td>1.46351</td>
<td>-0.482071</td>
<td>1.46622</td>
<td>0.00266468</td>
<td>0.00270099</td>
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<tr>
<td>0.2</td>
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<td>1.35557</td>
<td>-0.37658</td>
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<tr>
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<td>-0.263526</td>
<td>1.24763</td>
<td>-0.271089</td>
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<tr>
<td>0.4</td>
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<td>1.13969</td>
<td>-0.165999</td>
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</tr>
<tr>
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<td>1.04425</td>
<td>0.0124623</td>
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<tr>
<td>0.6</td>
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<tr>
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<td>0.815874</td>
<td>0.150873</td>
<td>0.833271</td>
<td>0.0173611</td>
<td>0.0173974</td>
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<tr>
<td>0.8</td>
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<td>0.256364</td>
<td>0.727781</td>
<td>0.0198105</td>
<td>0.0198468</td>
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<tr>
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<td>0.599994</td>
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<td>0.62229</td>
<td>0.0222599</td>
<td>0.0222962</td>
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<tr>
<td>1</td>
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<td>0.492054</td>
<td>0.467345</td>
<td>0.5168</td>
<td>0.0247093</td>
<td>0.0247457</td>
</tr>
</tbody>
</table>

**Table 1**

5 Conclusion

In this paper, we present a Theorem, that is show Nth-Order Fuzzy Differential equation (NOFDE) and the system ODE are equivalent, then we used suitable numerical methods for solving NOFDE.
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