Numerical solutions of fuzzy linear system differential equations and application of a Radioactivity decay model

Z. Gouyandeh 1*, A. Armand 1

(1) Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran

Copyright 2013 © Z. Gouyandeh and A. Armand. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, homotopy analysis method is presented for linear system of first-order fuzzy differential equations which involves a radioactivity decay model. We give the illustrative example to demonstrate the efficiency of the method. Finally, homotopy analysis method (HAM) is applied to obtain series solutions to the the radioactivity decay model.

Keywords: Linear system of first-order fuzzy differential equations, Homotopy analysis method, The Radioactivity decay model, Generalized Hukuhara differentiable

1 Introduction

The topic of Fuzzy Differential Equations, has been rapidly growing in recent years. The concept of the fuzzy derivative was first introduced by Chang and Zadeh [8], it was followed by many authors. The starting point of the topic in set valued differential equation and also fuzzy differential equation is Hukuhara’s paper [11]. The strongly generalized differentiability was defined by considering lateral H-derivatives [9]. Apparently the disadvantage of strongly generalized differentiability of a function in comparison H-differentiability is that, a fuzzy differential equation has no unique solution [6]. Recently, Stefanini and Bede by the concept of generalization of the Hukuhara difference for compact convex set [19], introduced generalized Hukuhara differentiability [18] for interval valued function and they shown that, this concept of differentiability have relationships with weakly generalized differentiability and strongly generalized differentiability. Many researcher have been suggested some analytical and numerical methods for solving fuzzy differential equations. (see e.g. [15, 4, 1, 3, 5, 2]).

In [7, 10, 12] authors transform an fuzzy differential equation to a system of ordinary differential equations. Base on this method, a non-homogenous n-dimensional system of first order linear fuzzy differential equation convert to a 2n-dimensional system of ordinary differential equations. Solaymani Fard and Ghal-Eh [16] used the variational iteration method to solve linear system of first-order fuzzy differential equations with fuzzy constant coefficients under Hukuhara differentiability.

In this study, we consider a class linear system of fuzzy differential equations under generalized Hukuhara differentiability. In addition, we apply the present method for Radioactivity decay model in continuous time. The mentioned model [16, 13] is given by

$$D_{gH}(x_1(t)) = -\lambda_1 \odot x_1(t) \oplus r$$
$$D_{gH}(x_2(t)) = \lambda_1 \odot x_1(t) \odot \lambda_2 \odot x_2(t)$$

(1.1)

*Corresponding author. Email address: zgouyandeh@yahoo.com, Tel:+989133331396
subject to
\[ x_1(0) = x_{10}, \quad x_2(0) = 0 \]

In this example, a target of stable nucleus (type 1) is placed in front of an accelerator beam. This leads to the production of a radioactive species of rate \( r \). The produced nuclei \( x_1(t) \) decay into unstable nuclei \( x_2(t) \) with decay constant \( \lambda_1 \).

Type 2 nuclei are unstable and decay with decay constant \( \lambda_2 \). It is assumed that \( r \) and \( x_1(t) \) suffer from uncertainty due to instrumental errors. Moreover, \( \lambda_1 \) and \( \lambda_2 \) may be determined from the theory of radioactive decay, which does not provide exact values. The values of \( r, x_1, x_2, \lambda_1 \) and \( \lambda_2 \) may be obtained from some group of experts. According to experts opinion, we may consider the smallest possible values, the highest possible values and the most likely values for \( r, x_1(0), \lambda_1 \) and \( \lambda_2 \). In other words, four triangular fuzzy numbers are constructed for these uncertain parameters [16].

The paper is organized as follows: we describe the basic notations and preliminaries in Section 2. In Section 3, the homotopy analysis method to approximate solution of nonlinear equation is presented. We give the illustrative example to clarify the details and efficiency of the method in Section 4 and in Section 5, we apply the present method for Radioactivity decay model. At the last section, we will have a conclusion of our study.

2 Basic preliminaries

2.1 Basic notations and definitions

In this section, we represent some definitions and introduce the necessary notation which will be used throughout the paper.

Denote \( \mathbb{R}_\mathcal{F} = \left\{ u : \mathbb{R}^n \to [0, 1] \mid u \text{ satisfies } (i) - (iv) \text{ below} \right\} \), where

(i) \( u \) is fuzzy convex;

(ii) \( u \) is normal, i.e., there exists an \( x_0 \in \mathbb{R}^n \) such that \( u(x_0) = 1 \);

(iii) \( u \) is upper semi-continuous;

(iv) closure of \( \{x \in \mathbb{R}^n \mid u(x) > 0\} \) is compact.

Then \( \mathbb{R}_\mathcal{F} \) is called the space of fuzzy numbers.

For \( 0 < \alpha \leq 1 \) denote \([u]^{\alpha} = \left\{ x \in \mathbb{R}^n \mid u(x) \geq \alpha \right\} = [u^\alpha_+, u^\alpha_-] \). Then from (i) to (iv), it follows that the \( \alpha \)-level set \([u]^{\alpha} \) is a closed interval for all \( \alpha \in [0, 1] \).

A triangular fuzzy number is defined as a fuzzy set in \( \mathbb{R}_\mathcal{F} \), that is specified by an ordered triple \( \tilde{u} = (a, b, c) \in \mathbb{R}^3 \) with \( a \leq b \leq c \) such that \( u^\alpha_+ = a + (b - a)\alpha \) and \( u^\alpha_- = c - (c - b)\alpha \) are the endpoints of \( \alpha \)-level sets for all \( \alpha \in [0, 1] \).

For arbitrary fuzzy number \([u]^{\alpha} = [u^\alpha_+, u^\alpha_-]\) and \([v]^{\alpha} = [v^\alpha_+, v^\alpha_-]\), we shall define addition, subtraction and multiplication as follows for \( 0 \leq \alpha \leq 1 \):

1. Addition: \([u]^{\alpha} + [v]^{\alpha} = [u^\alpha_+ + v^\alpha_-, u^\alpha_- + v^\alpha_+]\]

2. Subtraction: \([u]^{\alpha} - [v]^{\alpha} = [u^\alpha_+ - v^\alpha_-, u^\alpha_- - v^\alpha_+]\]

3. Multiplication: \([uv]^{\alpha} = \min\{u^\alpha_+v^\alpha_+, u^\alpha_+v^\alpha_-, u^\alpha_-v^\alpha_+, u^\alpha_-v^\alpha_-\}, \max\{u^\alpha_-v^\alpha_-, u^\alpha_+v^\alpha_-, u^\alpha_-v^\alpha_+, u^\alpha_+v^\alpha_+\}\]

If \( u \oplus v = w \), then \( w \ominus v = u \); here, \( \ominus \) is the Hukuhara difference.

**Definition 2.1.** ([18]) The generalized Hukuhara difference of two fuzzy numbers \( u, v \in \mathbb{R}_\mathcal{F} \) is defined as follows

\[ u \ominus_h v = w \iff \begin{cases} (i). \ u = v \oplus w; \\
(ii). v = u \oplus (-1)w. \end{cases} \]

Consider \([w]^{\alpha} = [w^\alpha_+, w^\alpha_-] \), then \( w^\alpha_- = \min\{u^\alpha_- - v^\alpha_-, u^\alpha_+ - v^\alpha_+\} \) and \( w^\alpha_+ = \max\{u^\alpha_- - v^\alpha_-, u^\alpha_+ - v^\alpha_+\} \).
Let parametric representation of fuzzy value function \( f : [a, b] \to \mathbb{R}^\mathcal{F} \) is expressed by \( f^\alpha(t) = [f^\alpha(t), f^\alpha(t)], t \in [a, b], \alpha \in [0, 1] .\)

**Definition 2.2.** ([18]) The generalized Hukuhara derivative of a fuzzy value function \( f : (a, b) \to \mathbb{R}^\mathcal{F} \) at \( t_0 \) is defined as

\[
f^\alpha_{gH}(t_0) = \lim_{h \to 0} \frac{f(t_0 + h) \ominus f(t_0)}{h}
\]

(2.2)

If \( f^\alpha_{gH}(t_0) \in \mathbb{R}^\mathcal{F} \) satisfying (2.2) exists, we say that \( f \) is generalized Hukuhara differentiable \((gH\text{-differentiable for short})\) at \( t_0 \).

Also we say that \( f(t) \) is \((i) - gH\text{-differentiable at } t_0 \) if

\[
(f^\alpha)'(t_0) = [(f^\alpha)'(t_0), (f^\alpha)'(t_0)]
\]

(2.3)

and \( f(t) \) is \((ii) - gH\text{-differentiable at } t_0 \) if

\[
(f^\alpha)'(t_0) = [(f^\alpha)'(t_0), (f^\alpha)'(t_0)].
\]

(2.4)

### 2.2 Linear system of fuzzy differential equations

Consider following linear system of first-order fuzzy differential equation

\[
DgH(\tilde{X}(t)) = A \odot \tilde{X}(t) \oplus \tilde{F}(t), \quad 0 \leq t \leq 1
\]

(2.5)

subject to

\[
\tilde{X}(0) = \tilde{X}_0
\]

where \( \tilde{A} = [a_{ij}]_{n \times n}, \tilde{X}(t) = [x_1(t), x_2(t), ..., x_n(t)], \tilde{F}(t) = [f_1(t), f_2(t), ..., f_n(t)], \tilde{X}_0 = [x_{10}, x_{20}, ..., x_{n0}] \) and also the elements of the coefficient matrix \( A \), the elements of the vectors \( \tilde{F}(t), X(t) \), \( X_0 \) are fuzzy numbers.

Let \( [x_m(t)]^\alpha = [(x_m^\alpha(t), x_m^\alpha(t))], m = 1, 2, ..., n \) are \( gH \)-differentiable such that no change in type of differentiability on \([a, b]\). If \( x_m(t) \) are \([i) - gH\text{-differentiable then } x_m'(t)]^\alpha = [(x_m^\alpha(t), x_m^\alpha(t))]. \) Thus, for \( m = 1, 2, ..., n \) Eq.(2.5) is transformed into the following system of ODEs:

\[
\begin{cases}
(x_m')^\alpha(t) = \sum_{j=1}^{n} a_{mj}^\alpha x_j^\alpha(t) + (f_m)^\alpha(t) \\
(x_m)^\alpha(0) = (x_{m0})^\alpha
\end{cases}
\]

(2.6)

and if \( x_m(t), m = 1, 2, ..., n \) are \([ii) - gH\text{-differentiable then } x_m'(t)]^\alpha = [(x_m^\alpha(t), x_m^\alpha(t))]. \) So

\[
\begin{cases}
(x_m')^\alpha(t) = \sum_{j=1}^{n} a_{mj}^\alpha x_j^\alpha(t) + (f_m)^\alpha(t) \\
(x_m)^\alpha(0) = (x_{m0})^\alpha
\end{cases}
\]

(2.7)

Where

\[
\begin{align*}
\frac{a_{mj}^\alpha}{f_m^\alpha} &= \min \{BU | B \in [(a_{mj})^\alpha, (a_{mj})^\alpha], U \in [(x_j)^\alpha(t), (x_j)^\alpha(t)]\} \\
\frac{a_{mj}^\alpha}{f_m^\alpha} &= \max \{BU | B \in [(a_{mj})^\alpha, (a_{mj})^\alpha], U \in [(x_j)^\alpha(t), (x_j)^\alpha(t)]\}
\end{align*}
\]

Accordingly, if \( x_m(t) \) are \([i) - gH\text{-differentiable, then the linear system (2.6)} \) can be interpreted in two different ways:
1. (i) If $A$ is a non-negative matrix,
\[
\begin{align*}
(x'_m)^a(t) &= \sum_{j=1}^{a} (a_{mj})^a (x_j)^a(t) + (f_m)^a(t) \\
(x_m)^a(0) &= (x_{m0})^a \\
(x'_m)^a(t) &= \sum_{j=1}^{a} (a_{mj})^a (x_j)^a(t) + (f_m)^a(t) \\
(x_m)^a(0) &= (x_{m0})^a
\end{align*}
\] (2.8)

2. (ii) If $A$ is a non-positive matrix,
\[
\begin{align*}
(x'_m)^a(t) &= \sum_{j=1}^{a} (a_{mj})^a (x_j)^a(t) + (f_m)^a(t) \\
(x_m)^a(0) &= (x_{m0})^a \\
(x'_m)^a(t) &= \sum_{j=1}^{a} (a_{mj})^a (x_j)^a(t) + (f_m)^a(t) \\
(x_m)^a(0) &= (x_{m0})^a
\end{align*}
\] (2.9)

3 Homotopy analysis method

In this section, we extended the applications of the homotopy analysis method to solve linear system differential equations. To achieve our goal, we consider the following nonlinear differential equations:
\[
\mathcal{N}_i[z_i(t)] = 0 \quad i = 1, 2, ..., n
\] (3.10)

3.1 Zeroth-order deformation equation

Liao [14], construct the so-called zeroth-order deformation equation:
\[
(1 - q)\mathcal{L}_i[\phi_i(t, q) - z_{0i}(t)] = qh_iH_i(t)\mathcal{N}_i[\phi_i(t, q)] \quad i = 1, 2, ..., n
\] (3.11)

where $q \in [0, 1]$ is an embedding parameter, $\mathcal{N}_i$ are nonlinear operators, $\mathcal{L}_i$ are auxiliary linear operators, $z_{0i}(t)$ are initial guesses satisfy the initial conditions, $h \neq 0$ are auxiliary parameters, $H_i(t) \neq 0$ are auxiliary functions and $\phi_i(t, q)$ are unknown functions.

It should be emphasized that one has great freedom to choose the initial guesses, the auxiliary linear operators $\mathcal{L}_i$, the auxiliary parameters $h_i$ and the auxiliary functions $H_i(t)$. When $q = 0$ and $q = 1$, we have from the zero-order deformation Eq.(3.11) that $\phi_i(t, 0) = z_{0i}(t)$ and $\phi_i(t, 1) = z_i(t)$ for $i = 1, 2, ..., n$.

Thus, as $q$ increases from 0 to 1, the solution $\phi_i(t, q)$ varies from the initial guess $z_{0i}(t)$ to the solution $z_i(t)$. Defining
\[
z_{im}(t) = \frac{1}{m!} \frac{\partial^m \phi_i(t, q)}{\partial q^m} \big|_{q=0} \quad i = 1, 2, ..., n
\] (3.12)

and expanding $\phi_i(t, q)$ in Taylor series with respect to the embedding parameter $q$, we have
\[
\phi_i(t, q) = z_{0i} + \sum_{m=1}^{\infty} z_{im}(t)q^m \quad i = 1, 2, ..., n
\] (3.13)

If the auxiliary linear operator, the initial guess, the auxiliary parameter $h$ and the auxiliary function $H_i(t)$ are properly chosen, the series Eq.(3.13) converges at $q = 1$. Then at $q = 1$, the series (3.13) becomes
\[
z_i(t) = z_{0i}(t) + \sum_{m=1}^{\infty} z_{im}(t) \quad i = 1, 2, ..., n
\] (3.14)
3.2 The mth-order deformation equation

Define the vector

$$\mathbf{Z}_{ij} = \{z_0(t), z_1(t), \ldots, z_i(t)\}, \quad i = 1, 2, \ldots, j$$

Differentiating the zero-order deformation Eq.(3.11) m times with respect to q, then setting q = 0 and finally dividing by m!, we have the mth-order deformation equation

$$\mathcal{L}_i[z_m(t) - \chi_mz_{(m-1)}(t)] = h_iH(t)c_m(\mathbf{Z}_{i(m-1)}(t)), \quad i = 1, 2, \ldots, n$$

(3.15)

where

$$c_m(\mathbf{Z}_{i(m-1)}(t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{L}_i(\phi(t,q))}{\partial q^{m-1}} \bigg|_{q=0}, \quad i = 1, 2, \ldots, n$$

(3.16)

and

$$\chi_m = \begin{cases} 0, & m \leq 1; \\ 1, & o.w. \end{cases}$$

The mth-order deformation equation (3.15) is linear and thus can be easily solved, especially by means of symbolic computation software Matlab, Mathematica and so on.

4 Numerical illustration

Consider the nonhomogeneous linear system of first-order fuzzy differential equations (2.5), with [16]

$$A^\alpha = \begin{pmatrix} [1 + 0.5\alpha, 2 - 0.5\alpha] & [0.5 + 0.5\alpha, 1.5 - 0.5\alpha] \\ [0.5 + 0.5\alpha, 1.5 - 0.5\alpha] & [2 + 0.5\alpha, 3 - 0.5\alpha] \end{pmatrix}$$

and

$$[f(t)]^0 = \begin{pmatrix} [1 + t + (0.5 + t)\alpha, 2 + 3t - (0.5 + t)\alpha] \\ [e^{-t} + \left(\frac{e^t - e^{-t}}{2}\right)\alpha, e^t - \left(\frac{e^t - e^{-t}}{2}\right)\alpha] \end{pmatrix}$$

for \(0 \leq t \leq 1\), and \(X^\alpha(0) = \begin{pmatrix} [1 + 0.5\alpha, 2 - 0.5\alpha] \\ [0.5\alpha, 1 - 0.5\alpha] \end{pmatrix}\). By considering \(x_1(t), x_2(t)\) are \([(i) - gH]-\)differentiable, the exact solutions of the problem for \(\alpha = 0\) obtained as follows

$$\begin{align*}
(x_1)^0(t) &= 0.35e^{2.20t} - 1.14t + 3.09e^{0.79t} + 0.08e^{-1.0t} - 2.53; \\
(x_2)^0(t) &= 0.28t + 0.85e^{2.20t} - 1.28e^{0.79t} - 0.34e^{-1.0t} + 0.77 \\
(x_1)^0(t) &= 1.54e^{4.08t} - 2.43t + 10.45e^{0.91t} - 6.03e^t - 4.03 \\
(x_2)^0(t) &= 1.25t + 2.13e^{4.08t} - 7.53e^{0.91t} + 4.02e^t + 2.45
\end{align*}$$

From Eq.(2.5), we have

$$\frac{d}{dt}X^\alpha(t) = A^\alpha X^\alpha(t) + f^\alpha(t)$$

(4.17)

So, we conclude that

$$\frac{d}{dt} \begin{pmatrix} (x_1)^\alpha(t) \\ (x_2)^\alpha(t) \end{pmatrix} = \begin{pmatrix} 1 + 0.5\alpha & 0.5 + 0.5\alpha & 0 & 0 \\ 0.5 + 0.5\alpha & 2 + 0.5\alpha & 0 & 0 \end{pmatrix} \begin{pmatrix} (x_1)^\alpha(t) \\ (x_2)^\alpha(t) \end{pmatrix} + \begin{pmatrix} 1 + t + (0.5 + t)\alpha \\ e^{-t} + \left(\frac{e^t - e^{-t}}{2}\right)\alpha \end{pmatrix}$$

$$+ \begin{pmatrix} 2 + 3t - (0.5 + t)\alpha \\ e^t - \left(\frac{e^t - e^{-t}}{2}\right)\alpha \end{pmatrix}$$
subject to the initial condition
\[ X^\alpha(0) = \begin{pmatrix} 1 + 0.5\alpha \\ 0.5\alpha \\ 2 - 0.5\alpha \\ 1 - 0.5\alpha \end{pmatrix} \text{ for all } \alpha \in [0, 1]. \]

Now, let us apply the procedure in Section 3 to obtain this approximate solution. According to the initial condition of Eq.(4.17), the solution can be expressed by a set of base functions
\[ \{e^{\alpha t}, \tau, e^{\alpha \tau}, e^{\alpha \tau^2}|n = 0, 1, \ldots\} \]
in the form
\begin{align*}
(x_1)_-(t) &= \sum_{n=0}^\infty d_{1n} e^{\alpha t}, \quad (x_2)_-(t) = \sum_{n=0}^\infty d_{2n} t \\
(x_1)_+(t) &= \sum_{n=0}^\infty d_{3n} e^{\alpha t}, \quad (x_2)_+(t) = \sum_{n=0}^\infty d_{4n} e^{\alpha t}
\end{align*}
where \( d_{ij} \) are coefficients to be determined. Now, we choose the linear operators
\[ \mathcal{L}_i[\phi^\alpha(t, q)] = \frac{\partial \phi^\alpha(t, q)}{\partial t} \]
possesses the property \( \mathcal{L}_i[C_i] = 0, i = 1, 2, 3, 4 \), where \( C_i \) are an integral constant to be determined by initial conditions. Thus, we obtain the \( m \)-th-order \( (m \geq 1) \) deformation equations
\[ \mathcal{L}_i[z_{im}^\alpha(t) - \chi_{m}z_{(m-1)}^\alpha(t)] = h_i H_i(t) \mathcal{R}_m(Z_{(m-1)}^\alpha), \quad i = 1, 2, 3, 4 \]
where
\begin{align*}
\mathcal{R}_{1m}(Z_{1(m-1)}^\alpha(t)) &= \frac{d}{dt}(x_1)_{-m-1}^\alpha(t) - (1 + 0.5\alpha)(x_1)_{-m-1}^\alpha(t) - (0.5 + 0.5\alpha)(x_2)_{-m-1}^\alpha(t) \\
&\quad - (1 - \chi_m)(1 + t + (0.5 + t)\alpha) \\
\mathcal{R}_{2m}(Z_{2(m-1)}^\alpha(t)) &= \frac{d}{dt}(x_2)_{-m-1}^\alpha(t) - (0.5 + 0.5\alpha)(x_1)_{-m-1}^\alpha(t) - (2 + 0.5\alpha)(x_2)_{-m-1}^\alpha(t) \\
&\quad - (1 - \chi_m)(e^{-t} + \left(\frac{e^t - e^{-t}}{2}\right)\alpha) \\
\mathcal{R}_{3m}(Z_{3(m-1)}^\alpha(t)) &= \frac{d}{dt}(x_1)_{-m-1}^\alpha(t) - (2 - 0.5\alpha)(x_1)_{-m-1}^\alpha(t) - (1.5 - 0.5\alpha)(x_2)_{-m-1}^\alpha(t) \\
&\quad - (1 - \chi_m)(2 + 3t - (0.5 + t)\alpha) \\
\mathcal{R}_{4m}(Z_{4(m-1)}^\alpha(t)) &= \frac{d}{dt}(x_2)_{-m-1}^\alpha(t) - (1.5 - 0.5\alpha)(x_1)_{-m-1}^\alpha(t) - (3 - 0.5\alpha)(x_2)_{-m-1}^\alpha(t) \\
&\quad - (1 - \chi_m)(e^{t} - \left(\frac{e^t - e^{-t}}{2}\right)\alpha)
\end{align*}
Now the solution of the \( m \)-th-order \( (m \geq 1) \) deformation equations becomes
\[ z_{im}^\alpha(t) = \chi_{m}z_{(m-1)}^\alpha(t) + h_i \int_0^t H_i(\tau) \mathcal{R}_m(Z_{(m-1)}^\alpha(t)) d\tau + C_i, \quad i = 1, 2, 3, 4, \quad m = 1, 2, \ldots \]
where the integration constants \( C_i(i = 1, 2, 3, 4) \) are determined by the initial conditions of Eq.(4.17). Then the series solutions expression by HAM can be written in the form
\begin{align*}
(x_1)_1^\alpha(t) &= \sum_{i=1}^m z_{1i}^\alpha(t), \quad (x_2)_1^\alpha(t) = \sum_{i=1}^m z_{2i}^\alpha(t) \\
(x_1)_2^\alpha(t) &= \sum_{i=1}^m z_{3i}^\alpha(t), \quad (x_2)_2^\alpha(t) = \sum_{i=1}^m z_{4i}^\alpha(t)
\end{align*}
for all \( \alpha \in [0, 1] \).

To influence of \( h_1 \) on the convergence of the Eqs.(4.19), we first plot the so-called h-curves of \(((x_1^0)'(0)\) and \(((x_1^0)''(0)\), as shown in Figure 2, it is easy to discover the valid region of \( h_1 \). Similarly, the values of \( h_i, i = 2, 3, 4 \) can be obtained.

Consider

\[
\begin{align*}
\zeta_{10}^0(t) &= (1 + 0.5\alpha)e^t, \\
\zeta_{20}^0(t) &= (0.5\alpha)t, \\
\zeta_{30}^0(t) &= (2 - 0.5\alpha)e^t, \\
\zeta_{40}^0(t) &= (1 - 0.5\alpha)e^t
\end{align*}
\]

as initial approximations and \( h_i = -1, H_i(t) = 1, i = 1, 2, 3, 4 \). In Table 1, the approximate solutions are compared with the exact solutions for \( m = 6 \) and in Figure 1, we compared the 6th-order approximation solution with exact solution in \( \alpha = 0, t = 0 \).

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x_1^0(t) ) Absolute Error</th>
<th>( x_2^0(t) ) Absolute Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.05</td>
<td>( 3.4950 \times 10^{-9} )</td>
<td>( 4.8545 \times 10^{-9} )</td>
</tr>
<tr>
<td>0.10</td>
<td>( 4.6259 \times 10^{-7} )</td>
<td>( 6.4179 \times 10^{-7} )</td>
</tr>
<tr>
<td>0.15</td>
<td>( 8.1728 \times 10^{-6} )</td>
<td>( 1.1338 \times 10^{-5} )</td>
</tr>
<tr>
<td>0.20</td>
<td>( 6.3343 \times 10^{-5} )</td>
<td>( 8.7882 \times 10^{-5} )</td>
</tr>
<tr>
<td>0.25</td>
<td>( 3.1265 \times 10^{-4} )</td>
<td>( 4.3377 \times 10^{-4} )</td>
</tr>
<tr>
<td>0.30</td>
<td>( 1.1603 \times 10^{-3} )</td>
<td>( 1.6098 \times 10^{-3} )</td>
</tr>
<tr>
<td>0.35</td>
<td>( 3.5374 \times 10^{-3} )</td>
<td>( 4.9078 \times 10^{-3} )</td>
</tr>
<tr>
<td>0.40</td>
<td>( 9.3406 \times 10^{-3} )</td>
<td>( 1.2959 \times 10^{-2} )</td>
</tr>
<tr>
<td>0.45</td>
<td>( 2.2103 \times 10^{-2} )</td>
<td>( 3.0665 \times 10^{-2} )</td>
</tr>
<tr>
<td>0.50</td>
<td>( 4.7976 \times 10^{-2} )</td>
<td>( 6.6562 \times 10^{-2} )</td>
</tr>
</tbody>
</table>

Table 1: The HAM approximation errors of numerical illustration

Figure 1: Graph of the HAM approximation error of Numerical illustration for \( x_1^0(0) \) (left) and \( x_2^0(0) \) (right)
From Eq. (2.5), we have

\[ \alpha \]

The exact solutions of this problem for \( x(t) \) subject to the initial condition

\[ e \]

where \( e \)

subject to initial conditions \( x(0) = (x_1, x_2, \ldots, x_n) \) and \( \tilde{x}(0) = (0, 0, 0) \).

Now, let \( x(0) = (995, 1000, 1005) \), \( \tilde{r} = (4.9, 5.5, 1.1) \), \( \tilde{\lambda} = (0.2, 0.3, 0.4) \) and \( r = (0.02, 0.03, 0.04) \). Moreover, consider \( x(t), x(t) \) are \((i-gH)\)-differentiable.

From Eq. (2.5), we have

\[ \frac{d x}{dt} = A x(t) + f(t) \]

where

\[ X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & -0.4 + 0.1 \alpha \\ 0.2 + 0.1 \alpha & 0 & 0.04 + 0.01 \alpha \\ -0.2 - 0.1 \alpha & 0.4 - 0.1 \alpha & -0.02 - 0.01 \alpha \end{pmatrix}, \quad f(t) = \begin{pmatrix} 4.9 + 0.1 \alpha \\ 0 \\ 5.1 - 0.1 \alpha \end{pmatrix} \]

subject to the initial condition \( x(0) = \begin{pmatrix} 995 + 5 \alpha \\ 0 \\ 1005 - 5 \alpha \end{pmatrix} \) for all \( \alpha \in [0, 1] \).

The exact solutions of this problem for \( \alpha = 1 \) by [17] are

\[
\begin{align*}
(x_1(t) &= \frac{500}{3} + \frac{2950}{3} e^{\frac{5}{3} t} \\
(x_2(t) &= \frac{5}{3} - 885 e^{\frac{-5}{3} t} + 115 \sqrt{10} \sin \left( \frac{3 \sqrt{10} t}{100} \right) + \frac{2650}{3} \cos \left( \frac{3 \sqrt{10} t}{100} \right) \\
(x_3(t) &= \frac{500}{3} + 2950 e^{\frac{-5}{3} t} - 1150 \cos \left( \frac{3 \sqrt{10} t}{100} \right) + \frac{2650}{3} \sin \left( \frac{3 \sqrt{10} t}{100} \right)
\end{align*}
\]

\[ \]

5 Application of the Radioactivity decay model

In this section, we will apply the method presented for a stable Radioactivity decay model [16]. In this application, we investigate \( 0 \in R, \). Then, the model (1.1) converts to the problem

\[
\frac{d}{dt} \begin{pmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{pmatrix} = \begin{pmatrix} -\tilde{\lambda}_1 & 0 \\ \tilde{\lambda}_1 & -\tilde{\lambda}_2 \end{pmatrix} \begin{pmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{pmatrix} + \begin{pmatrix} \tilde{r} \\ 0 \end{pmatrix}
\]

subject to initial conditions \( \tilde{x}(0) = (x_{1f}, x_{2f}, \ldots, x_{nf}) \) and \( \tilde{x}(0) = (0, 0, 0) \).

Figure 2: The h-curves for Numerical illustration, solid line: 6th-order approximation of \((x_1')_1(0)\); dashed line: 6th-order approximation of \((x_1')_2(0)\).
The computational results by choose the initial approximations $Z^{(0)}_{\alpha i}(0) = \begin{pmatrix} 995 + 5\alpha \\ 0 \\ 1005 - 5\alpha \end{pmatrix}$ and the linear operator as in (4.18) and $h_i = -1, H(t) = 1$ have been reported in Table (2). By attention to Figure 3, we could take another values for $h$ except 1.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x'_1(t)$ Absolute Error</th>
<th>$x'_2(t)$ Absolute Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>0.10</td>
<td>$9.9123 \times 10^{-10}$</td>
<td>$9.9257 \times 10^{-10}$</td>
</tr>
<tr>
<td>0.20</td>
<td>$6.3177 \times 10^{-8}$</td>
<td>$5.6917 \times 10^{-8}$</td>
</tr>
<tr>
<td>0.30</td>
<td>$7.1658 \times 10^{-7}$</td>
<td>$6.4557 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.40</td>
<td>$4.0092 \times 10^{-6}$</td>
<td>$3.6119 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.50</td>
<td>$1.5229 \times 10^{-5}$</td>
<td>$1.3720 \times 10^{-5}$</td>
</tr>
<tr>
<td>0.60</td>
<td>$4.5283 \times 10^{-5}$</td>
<td>$4.0797 \times 10^{-5}$</td>
</tr>
<tr>
<td>0.70</td>
<td>$1.1371 \times 10^{-4}$</td>
<td>$1.0244 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.80</td>
<td>$2.5231 \times 10^{-4}$</td>
<td>$2.2731 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.90</td>
<td>$5.0937 \times 10^{-4}$</td>
<td>$4.5891 \times 10^{-4}$</td>
</tr>
<tr>
<td>1.00</td>
<td>$9.5450 \times 10^{-4}$</td>
<td>$8.5995 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

Table 2: Numerical results for the radioactivity decay model.

Figure 3: The $h_1$-curve for the radioactivity decay model, solid line:6th-order approximation of $(x'_1)_{(0)}$; dashed line: 6th-order approximation of $(x''_1)_{(0)}$.

6 Conclusion

In this Letter, the homotopy analysis method (HAM) has been applied to obtain the solution of a nonhomogeneous system of first-order linear fuzzy differential equations under generalized Hukuhara differentiability. HAM provided us with a convenient way to control the convergence of approximation series, which is a fundamental qualitative difference in analysis between HAM and other methods.
References


http://dx.doi.org/10.1016/j.ins.2013.07.028

http://dx.doi.org/10.1016/j.na.2008.12.005

http://dx.doi.org/10.1007/978-3-540-85027-4_25